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Emil J. Straube

Lectures on the $\mathcal{L}^2$-Sobolev Theory of the $\overline{\partial}$-Neumann Problem

European Mathematical Society

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Preface

In the summer and fall of 2005, I gave a series of lectures at the Erwin Schrödinger International Institute of Mathematical Physics in Vienna on the basic $\mathcal{L}^2$-Sobolev theory of the $\bar{\partial}$-Neumann problem on domains in $\mathbb{C}^n$. These lectures, aimed at advanced graduate students and young researchers, produced a set of notes of about eighty pages. In the spring semester of 2006, I taught a graduate course at Texas A&M University on the same topic, revising and somewhat expanding the notes in the process. Subsequently, I set out to further revise the notes and to make them more or less self-contained, as far as the $\bar{\partial}$-Neumann problem is concerned. The intent remained the same: to provide a thorough introduction to the topic in the title, in the setting of bounded pseudoconvex domains in $\mathbb{C}^n$, that leads up to current research. This monograph is the result.

The basic $\mathcal{L}^2$-theory is presented in Chapter 2. In Chapter 3 we discuss the subelliptic estimates on strictly pseudoconvex domains. From the point of view of leading up to current research, Chapter 4 on compactness and Chapter 5 on regularity in Sobolev spaces are the most important. For a detailed description of the contents of these chapters, along with historical remarks, I refer the reader to the introductory Chapter 1.

A word about prerequisites. The reader is assumed to have a solid background in basic complex and functional analysis (including the elementary $\mathcal{L}^2$-Sobolev theory and distributions). Some knowledge in several complex variables is clearly helpful, if only for motivation. Concerning partial differential equations, not much is assumed. The elliptic regularity of the Dirichlet problem for the Laplacian is quoted a few times. On the other hand, the ellipticity results needed for elliptic regularization in Chapter 3 are proved from scratch.

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Some of the results presented here come from my own work. For the most part, these results were obtained in collaboration with others. I would like to take this opportunity to thank my coauthors over the years for sharing their ideas.

College Station, Texas | Emil Straube
December 2009

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1 Introduction

In this chapter[1], we give a brief historical introduction to the $\bar{\partial}$-Neumann problem, combined with an outline of the organization of this monograph.

The $\bar{\partial}$-Neumann problem was formulated in the fifties by D. C. Spencer as a means to generalize the theory of harmonic integrals, i.e., Hodge theory, to non-compact complex manifolds. Apart from antecedents (such as [120], [146], [147]), the only written record of this introduction appears to be a set of notes [279] of lectures given at the Collège de France in 1955 ([199], p. 19). But then 'the early work on the $\bar{\partial}$-Neumann problem owes much more to D. C. Spencer than is documented in print' ([173], p. 330). For domains in $\mathbb{C}^n$, which is the context we restrict ourselves to in this monograph, the problem can be formulated as follows. Denote by Ω a pseudoconvex domain in $\mathbb{C}^n$, and by $\mathcal{L}^2_{(0,q)}(\Omega)$ the space of $(0,q)$-forms on Ω with square integrable coefficients. Each such form u can be written uniquely as a sum

$$u = {\sum_J}' u_J \, d\bar{z}_J, \tag{1.1}$$

where $J = (j_1, \dots, j_q)$ is a multi-index with $j_1 < j_2 < \cdots < j_q$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, and the $'$ indicates summation over increasing multi-indices. The inner product

$$(u, v) = \Big({\sum_J}' u_J \, d\bar{z}_J, {\sum_J}' v_J \, d\bar{z}_J \Big) = {\sum_J}' \int_\Omega u_J \overline{v_J} \, dV \tag{1.2}$$

turns $\mathcal{L}^2_{(0,q)}(\Omega)$ into a Hilbert space. Set

$$\bar{\partial}\Big({\sum_J}' u_J \, d\bar{z}_J \Big) = \sum_{j=1}^{n} {\sum_J}' \frac{\partial u_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_J, \tag{1.3}$$

where the derivatives are computed as distributions, and the domain of $\bar{\partial}$ is defined to consist of those $u \in \mathcal{L}^2_{(0,q)}(\Omega)$ where the result is a $(0, q+1)$-form with square integrable coefficients. Then $\bar{\partial} = \bar{\partial}_q$ is a closed, densely defined operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to $\mathcal{L}^2_{(0,q+1)}(\Omega)$, and as such has a Hilbert space adjoint. This adjoint is denoted by $\bar{\partial}^*_q$. (We will not use the subscripts when the form level at which the operators act is clear or not an issue.) One can check that $\bar{\partial}\bar{\partial} = 0$, so that we arrive at a complex, the $\bar{\partial}$ (or Dolbeault)-complex:

$$\mathcal{L}^2(\Omega) \xrightarrow{\bar{\partial}} \mathcal{L}^2_{(0,1)}(\Omega) \xrightarrow{\bar{\partial}} \mathcal{L}^2_{(0,2)}(\Omega) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{L}^2_{(0,n)}(\Omega) \xrightarrow{\bar{\partial}} 0.$$

In analogy to the Laplace–Beltrami operator associated to the de Rham complex on a Riemannian manifold, one forms the complex Laplacian

$$\Box_q = \bar{\partial}_{q-1}\bar{\partial}^*_{q-1} + \bar{\partial}^*_q \bar{\partial}_q, \tag{1.4}$$

[1]This chapter is a modified and expanded version of the introduction to my survey [286]. I am grateful to K. Diederich and J. Kohn for comments regarding that introduction.

with domain so that the compositions are defined. Alternatively, $\Box_q$ can be defined as the (unique) self adjoint operator on $\mathcal{L}^2_{(0,q)}(\Omega)$ associated to the quadratic form $Q(u,u) = \|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2$. The $\bar{\partial}$-Neumann problem is the problem of inverting $\Box_q$; that is, given $v \in \mathcal{L}^2_{(0,q)}(\Omega)$, find $u \in \mathrm{Dom}(\Box_q)$ such that $\Box_q u = v$. Note that $\mathrm{Dom}(\Box_q)$ involves the two boundary conditions $u \in \mathrm{Dom}(\bar{\partial}^*)$ and $\bar{\partial}u \in \mathrm{Dom}(\bar{\partial}^*)$; these are the $\bar{\partial}$-Neumann boundary conditions. The condition $u \in \mathrm{Dom}(\bar{\partial}^*)$ is equivalent to a Dirichlet condition for the (complex) normal component of u. Similarly, the condition $\bar{\partial}u \in \mathrm{Dom}(\bar{\partial}^*)$ is equivalent to a Dirichlet condition on the normal component of $\bar{\partial}u$, that is, a complex Neumann condition for u. The two conditions together are referred to as the $\bar{\partial}$-Neumann boundary conditions.

From the point of view of partial differential equations, the $\bar{\partial}$-Neumann problem represents the prototype of a problem where the operator is elliptic, but the boundary conditions are not coercive (so that the classical elliptic theory does not apply). From the point of view of several complex variables, the importance of the problem stems from the fact that its solution provides a Hodge decomposition in the context of the $\bar{\partial}$-complex, together with the attendant elegant machinery (as envisioned by Spencer). For example, such a decomposition readily produces a solution to the inhomogeneous $\bar{\partial}$ equation, as follows. Assume for the moment that $\Box_q$ has a (bounded) inverse in $\mathcal{L}^2_{(0,q)}(\Omega)$, say N_q. Then we have the orthogonal decomposition

$$u = \bar{\partial}\bar{\partial}^* N_q u + \bar{\partial}^*\bar{\partial} N_q u, \quad u \in L^2_{(0,q)}(\Omega). \tag{1.5}$$

If $\bar{\partial}u = 0$, then $\bar{\partial}^*\bar{\partial}N_q u$ is $\bar{\partial}$-closed as well (from (1.5)). Consequently, $\bar{\partial}^*\bar{\partial}N_q u = 0$ (since it is also orthogonal to $\ker(\bar{\partial})$), and

$$u = \bar{\partial}(\bar{\partial}^* N_q u), \tag{1.6}$$

with $\|\bar{\partial}^* N_q u\|^2 = (\bar{\partial}\bar{\partial}^* N_q u, N_q u) \le C\|u\|^2$. Thus the operator $\bar{\partial}^* N_q$ provides an $\mathcal{L}^2$-bounded solution operator to $\bar{\partial}$. In fact, this operator gives the (unique) solution orthogonal to $\ker(\bar{\partial})$ (equivalently: the solution with minimal norm). This solution is called the canonical solution.

That $\Box_q$ *does* have a bounded inverse N_q was known for strictly pseudoconvex domains by the early 1960s. Kohn ([187], [189], [188], [190]), starting from his generalization of an estimate discovered by Morrey ([228]), showed that in this case, not only is there an $\mathcal{L}^2$-bounded inverse, but N_q exhibits a subelliptic gain of one derivative as measured in the $\mathcal{L}^2$-Sobolev scale. Another interesting approach was given by the second author in [229], [230]. Shortly after Kohn's work, Hörmander ([170], [172], see also Andreotti–Vesentini [2] for similar techniques), combining ideas from [228], [188], and [4] with the use of weighted norms, proved certain Carleman type estimates which in the case of bounded pseudoconvex domains imply the existence of N_q as a bounded self-adjoint operator on $\mathcal{L}^2_{(0,q)}(\Omega)$. The weights are such that these techniques are also applicable in the $\mathcal{L}^2_{\mathrm{loc}}(\Omega)$-category. Interior elliptic regularity, applied to the weighted canonical solution, then gives (a new proof of) solvability of $\bar{\partial}$ in $C^\infty(\Omega)$ as well[2]. Other early applications of the new ideas included the real analytic embedding of compact real analytic manifolds ([228], [229], [230])[3], a new solution ([188], [170]) of the Levi problem[4], a new proof ([188]) of the Newlander–Nirenberg theorem on integrable almost complex structures ([234]), and, in general, an approach to several complex variables which takes advantage of $\bar{\partial}$-methods ([170], [172], [125], [243]). Interesting 'eyewitness' accounts of this foundational period by two of the principals appear in [173] and [199], respectively.

We reverse the historical order and discuss the $\mathcal{L}^2$-results on general pseudoconvex domains in Chapter 2 and the subelliptic estimates on strictly pseudoconvex domains in Chapter 3. The Carleman type estimates of Hörmander and Andreotti–Vesentini arise from considering the $\bar{\partial}$-complex in weighted $\mathcal{L}^2$-norms. This point of view results in a very useful extra term in the so called Kohn–Morrey formula (the basis for the results in the strictly pseudoconvex case) when the weights have certain plurisubharmonicity properties. A little over twenty years later, Ohsawa and Takegoshi ([244]) discovered that also introducing a 'twisting' factor into the $\bar{\partial}$-complex results in yet another additional new term which allows to compensate, in certain situations, for the lack of plurisubharmonicity in the weights. Their work was simplified and extended in the mid nineties by Berndtsson ([35]), McNeal ([220]), and Siu ([275]). Boas and the author then noted ([52]) that it is advantageous to base the $\mathcal{L}^2$-existence theory on the resulting 'twisted' version of the Kohn–Morrey–Hörmander formula as well. We take this approach in Chapter 2. The chapter closes with an application to extension, with $\mathcal{L}^2$-bounds, of holomorphic functions from affine submanifolds: we prove (the most basic version of) the Ohsawa–Takegoshi extension theorem.

It is not hard to see that Kohn's results for strictly pseudoconvex domains are optimal: N can never gain more than one derivative, and it can gain one derivative only when the domain is strictly pseudoconvex. However, under what circumstances subellipticity with a fractional gain of less than one derivative holds was not understood until the early eighties. Kohn gave sufficient conditions in [193] and noted that work of Diederich and Fornæss ([107]) implies that these conditions are satisfied when the boundary is real analytic. Kohn's students Catlin ([66], [67], [70]) and D'Angelo ([87], [88], [89]) resolved the issue: on a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, the

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[2] For domains of holomorphy, this existence theorem was obtained in the early fifties via sheaf theoretic methods (Cartan [61], Serre [267], Dolbeault [118]). The solution of the Levi problem (the fact that pseudoconvex domains are domains of holomorphy, see below), also accomplished by the early fifties, then implies solvability on pseudoconvex domains. The remarks following the proof of Theorem 2.14 contain further details.

[3] Shortly after [228] was circulated, this result was generalized by Grauert ([149]), using sheaf theoretic methods, to manifolds with countable topology. Moreover, [228] contains a gap (fixed by the author in [229], [230], see also [189], [188], [170]) related to density in the graph norm of forms smooth up to the boundary (see Proposition 2.3; for historical details, see [173]). Our discussion shows that nevertheless, [228] was quite influential.

[4] The Levi problem, that is, the question whether pseudoconvex domains are domains of holomorphy, was one of the main problems in several complex variables during the first half of the last century. It was solved in the affirmative independently by Bremermann ([58]), Norguet ([238]), and Oka ([246], [247]). See Remarks (ii) and (iii) following the proof of Theorem 2.16 and Remark (i) following the proof of Theorem 3.7 for details and references, both historical and mathematical.

$\bar{\partial}$-Neumann problem is subelliptic if and only if each boundary point is of finite type, that is, the order of contact, at the point, of complex varieties with the boundary is bounded above. This elegant characterization not withstanding, the question of how to determine the exact range of subellipticity remains open. One of the main tools in [193] is provided by the notion of subelliptic multipliers and their ideals. Although subellipticity is established in [70] by a different method, these ideas raised algebro-geometric questions of independent interest (see [236], [278] for recent work), and they turned out to be influential in later developments in complex and algebraic geometry ([232], [99], [276], [277], [198]).

We take up the subelliptic estimates for strictly pseudoconvex domains in Chapter 3. Because $\Box_q$ acts coefficientwise as (a constant multiple of) the real Laplacian, and in view of the boundary term in the Kohn–Morrey formula, the estimates at the ground level are a consequence of the corresponding Sobolev estimates in the Dirichlet problem for the real Laplacian. Lifting these estimates to higher Sobolev norms leads to a situation common in partial differential equations: one can prove certain Sobolev estimates, *assuming* that the Sobolev norms in question are finite (because one has to absorb these norms). But that these norms are finite is precisely what one wants to prove. The classical method to deal with this problem consists in obtaining uniform estimates for difference quotients (which *are* in $\mathcal{L}^2$ if the function/form is), and then letting the difference parameter tend to zero, rather than estimating derivatives directly. A method better suited for the $\bar{\partial}$-Neumann problem is elliptic regularization, developed in the context of operators defined by certain quadratic forms by Kohn and Nirenberg in the mid sixties ([201]). We give a careful discussion of the method in the case of the $\bar{\partial}$-Neumann problem and prove in particular that the regularized operators do have the claimed elliptic properties. Thus Chapter 3 gives an essentially self-contained proof of the subelliptic estimates in the strictly pseudoconvex case. By contrast, the general subelliptic estimates on domains of finite type are only briefly described, and subelliptic multipliers are omitted entirely; a detailed treatment of each of these topics would warrant a monograph in its own right.

When N_q does not gain derivatives, but is still compact (as an operator on $\mathcal{L}^2_{(0,q)}(\Omega)$), it follows from the already quoted work of Kohn and Nirenberg ([201]) in the mid sixties that N_q preserves the Sobolev spaces $W^s_{(0,q)}(\Omega)$ for all $s \geq 0$. In particular, N_q preserves $C^\infty_{(0,q)}(\overline{\Omega})$ (it is globally regular). These two authors did not, however, investigate when the compactness condition is actually satisfied. But work of Catlin ([68], compare also Takegoshi [293]) and Sibony ([272]) in the eighties shows that compactness provides indeed a viable route to global regularity: the compactness condition can be verified on large classes of domains. This verification was achieved via a potential theoretic condition called property(P), introduced and shown to imply compactness in [68]. In [272], property(P) is studied in detail using tools from Choquet theory. One striking result is that even when the set of boundary points of infinite type is large, for example has positive measure, property(P), and hence compactness, may still hold. We prove these results in Chapter 4.

The most blatant violation of property(P) is an analytic disc in the boundary. The obvious question whether such a disc is necessarily an obstruction to compactness received an affirmative answer quickly for the case of domains in $\mathbb{C}^2$ (commonly attributed to unpublished work of Catlin which became folklore). When the domain is in $\mathbb{C}^n$ with $n \geq 3$, the answer is not known. Şahutoğlu and the author recently obtained a partial answer which does generalize the $\mathbb{C}^2$ result to higher dimensions ([265]): when the disc contains a point at which the domain is strictly pseudoconvex in the directions transverse to the disc (a condition void in $\mathbb{C}^2$), then this disc is indeed an obstruction to compactness. On the other hand, Christ's student Matheos was able to show in his dissertation ([217]) in the mid nineties that there are obstructions to compactness more subtle than discs in the boundary. Shortly afterwards, Fu and the author discovered that there is however a large class of domains where the analysis, the potential theory, and the geometry mesh perfectly ([141]). They proved that on a locally convexifiable domain, the following three conditions are equivalent: the $\bar{\partial}$-Neumann operator is compact; the boundary satisfies property(P); the boundary contains no analytic discs.

According to a recent result of Christ and Fu ([86]), compactness and property(P) are equivalent also on smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$. (By what was said above, on these domains, they imply the absence of analytic discs from the boundary, but are not equivalent to it.) In general, however, it is not understood how much room there is between compactness and property(P). In fact, until about five years ago, the only way to obtain compactness of the $\bar{\partial}$-Neumann operator was via verifying P. Then, in [285], [231], the authors developed a new method for verifying compactness in certain cases. But while the method does not proceed via property(P), it is not clear whether among the domains where it applies, there are ones without property(P). A little earlier, McNeal had introduced a modification of property(P) that is formally weaker and that still implies compactness ([221]). To what extent it is strictly weaker is not understood at present. These results (with the exception of [86]) are also proved in Chapter 4.

Studying regularity properties of a differential operator is natural from a partial differential equations perspective. When the $\bar{\partial}$-Neumann operator N_q is globally regular, that is, when it preserves $C^{\infty}_{(0,q)}(\bar{\Omega})$, the canonical solution operator $\bar{\partial}^* N_q$ gives a solution operator for $\bar{\partial}$ which preserves $C^{\infty}(\bar{\Omega})$ as well. There are other important implications of global regularity for several complex variables; chief among these is the relevance for boundary behavior of biholomorphic or proper holomorphic maps. Work of Bell, Catlin, Diederich, Fornæss, and Ligocka ([32], [23], [30], [108]) shows that if Ω_1 and Ω_2 are two bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries, such that the $\bar{\partial}$-Neumann operator on $(0,1)$-forms on Ω_1 is globally regular[5], then any proper holomorphic map from Ω_1 to Ω_2 extends smoothly to the boundary of Ω_1.

[5]Actually, the regularity property that is needed in these results is global regularity of the Bergman projection, which in the pseudoconvex case is a consequence of global regularity of the $\bar{\partial}$-Neumann operator (more can be said; see Theorem 5.5 for the precise relationship). [32], which deals with biholomorphic maps, does not require pseudoconvexity; for a generalization in the case of proper maps to the nonpseudoconvex setting, see [25].

This result represents a vast generalization (as well as a simplification) of the celebrated mapping theorem of Fefferman ([123]), which covered strictly pseudoconvex domains and biholomorphic maps. It is highly nontrivial: in contrast to the one variable situation, where at least the biholomorphic case is classical (going as far back as Painlevé ([249], [250]; see [31] for the history), the general case in higher dimensions, even for biholomorphic maps, is open. Further exposition of the ideas and issues involved here can be found in [113], [16], [90], [31], [27], [206], [131], [81], [139].

In the early seventies, Kohn ([191]) noticed that by choosing suitable weights in Hörmander's method, one can obtain a weighted $\bar{\partial}$-Neumann operator that is continuous in Sobolev norms up to a certain level. More precisely, for every $k \in \mathbb{N}$, there is a weight so that the weighted $\bar{\partial}$-Neumann operator is continuous on $W^s_{(0,q)}(\Omega)$ for $0 \leq s \leq k$. The associated canonical solution operator is also continuous in $W^s(\Omega)$, $0 \leq s \leq k$. When combined with a Mittag-Leffler argument (credited in [192] to Hörmander), these solution operators yield a solution of $\bar{\partial}$ in the $C^\infty(\overline{\Omega})$-category on any smooth bounded pseudoconvex domain. The weighted theory also allows one to determine the exact relationship, discovered in the late eighties ([47]), between regularity properties of the $\bar{\partial}$-Neumann operators and those of the Bergman projections. Chapter 5 begins with these results.

Global regularity may hold when compactness fails. Throughout the eighties and into the early nineties, there appeared a series of results on global regularity that concerned domains with transverse symmetries ([28], [8], [10], [281], [79]), domains with partially transverse symmetries that allow the normal to be well approximated by holomorphic fields on the rest of the boundary ([46], [49]), or that combined these techniques on a portion of the boundary with subellipticity or compactness arguments on the rest of the boundary ([78], [44]). These methods apply in particular to Reinhardt domains and to many circular domains.

In the early nineties, Boas and the author proved in [48] that if Ω admits a defining function whose complex Hessian is positive semi-definite at points of the boundary (a condition slightly more restrictive than pseudoconvexity), then the $\bar{\partial}$-Neumann problem is globally regular (for all q). This class of domains includes in particular all (smooth) convex domains. (For convex domains in $\mathbb{C}^2$, the result was obtained independently by Chen, [80].) The proof is based on the existence of certain families of vector fields which have good approximate commutator properties with $\bar{\partial}$ (different treatments were given recently in [169], [287]). This method also covers (and was inspired by) the results mentioned earlier based on transverse symmetries and holomorphic vector fields. When computing the relevant commutators, there is a one-form, introduced into the literature by D'Angelo ([91], [92]), that comes up naturally. The existence of the required families of vector fields is equivalent to this one form being 'approximately exact'. From this point of view, the case when the domain admits a defining function that is plurisubharmonic at the boundary is the 'trivial' case: the form vanishes (in the directions that matter), so it is trivially approximately exact.

The same authors then studied the situation when the boundary points of infinite type form a complex submanifold (with boundary) of the boundary of the domain

([51]). The form mentioned in the previous paragraph defines a de Rham cohomology class on such a submanifold. This cohomology class is the obstruction to the existence of the vector fields needed. In particular, a simply connected complex manifold in the boundary is benign for global regularity of the $\bar{\partial}$-Neumann problem. The obvious question whether the cohomology class is also necessarily an obstruction to global regularity is still open. It is noteworthy that this class also plays a role in deciding whether or not the closure of the domain admits a Stein neighborhood basis; in this role, it had appeared already in the late seventies in work of Bedford and Fornæss ([17]).

A natural next step was taken in [289], where the authors considered the case where the boundary is finite type except for a Levi-flat piece which is 'nicely' foliated by complex hypersurfaces. Whether or not the families of vector fields with good approximate commutator properties with $\bar{\partial}$ exist turns out to be equivalent to a property of the Levi foliation much studied in foliation theory, namely whether or not the foliation can be defined *globally* by a closed one-form. These connections, while of interest in their own right, also allow one to bring tools from foliation theory to bear on the problem of finding the required families of vector fields and thus obtaining global regularity of the $\bar{\partial}$-Neumann operator ([289], [132]).

The question how to unify the two main approaches to global regularity, via compactness or via vector fields with good commutator properties with $\bar{\partial}$, arose as soon as [48], [51] were completed. Chapter 5 closes with a recent result of the author that proposes such a unified treatment of global regularity.

So far we have only discussed positive results. Whether global regularity holds on general pseudoconvex domains turned out to be a very difficult question that was resolved only in the mid nineties. Barrett ([11], see also [9] and [183] for predecessors) showed that on the worm domains of Diederich and Fornæss ([105]), N_1 does not preserve $W^s_{(0,1)}(\Omega)$ for s sufficiently large, depending on the winding (that is, exact regularity fails). Christ ([83], see also [84], [85]) resolved the question by proving certain a priori estimates for N_1 on these domains that would imply exact regularity in Sobolev spaces (and thus would contradict Barrett's result) if N_1 were to preserve the space of forms smooth up to the boundary. While worm domains are discussed in Chapter 5, the reader is referred to the original sources for the proofs of the Barrett-Christ results.

In addition to these proofs, and a detailed treatment of subelliptic estimates already mentioned, there are other important topics in, or closely related to, the $\mathcal{L}^2$-Sobolev theory on bounded domains in $\mathbb{C}^n$ that are not treated in this monograph. The spectral theory of the $\bar{\partial}$-Neumann operator studies connections between the spectrum and the boundary geometry; compare [137], [138], [140] and the references there. One can also ask what happens with regard to Sobolev estimates when the domain is not assumed to be C^∞-smooth. For results on Lipschitz domains, we refer the reader to Shaw's survey [270]. It is furthermore useful to consider (a version of) the $\bar{\partial}$-Neumann problem on nonpseudoconvex domains. When $q > 1$, an assumption weaker than pseudoconvexity suffices to make the $\mathcal{L}^2$-Hilbert space machine run, starting from the (twisted) Kohn–Morrey–Hörmander formula. While we do discuss this case, there are variants of the

Kohn–Morrey–Hörmander formula still more general that we do not include. In all these cases, results typically hold for a restricted set of form levels q, the restrictions depending on properties of the boundary related to pseudoconvexity (see for example [171], [125], [268], [305], [169], [1], [255], [271]). Another topic closely related to the subject of this monograph, but not treated here, is the $\mathcal{L}^2$-Sobolev theory of the boundary complex, or, more generally, of the $\bar{\partial}_b$-complex on CR-manifolds ([269], [45], [195], [81], [197], [185], [235], [200], [255], [254]). Finally, we mention recent activity towards creating an $\mathcal{L}^2$-theory of $\bar{\partial}$ on singular complex spaces ([109], [129], [248], [262] and their references).

2 The $\mathscr{L}^2$-theory

2.1 The basic $\mathscr{L}^2$-setup

Unless otherwise specified, Ω will be a *bounded* domain in $\mathbb{C}^n$, u a $(0,q)$-form on Ω. That is

$$\begin{aligned} u(z) &= {\sum_{|J|=q}}' u_J(z)\, d\bar{z}_J, \\ J &= (j_1, j_2, \ldots, j_q), \quad 1 \le j_1 < j_2 < \cdots < j_q \le n, \\ d\bar{z}_J &= d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \cdots \wedge d\bar{z}_{j_q}. \end{aligned}$$

Here, the coefficients $u_J(z)$ are functions (belonging to various function classes) on Ω. We take these coefficients to be defined for all multi-indices J of length q, by the requirement that they are antisymmetric (interchanging two indices results in multiplication by -1). $\mathscr{L}^2_{(0,q)}(\Omega)$ consists of the $(0,q)$-forms u such that

$$\|u\|^2 := {\sum_{|J|=q}}' \int_\Omega |u_J(z)|^2\, dV(z) < \infty$$

with associated inner product

$$(u, v) = \Big({\sum_{|J|=q}}' u_J\, d\bar{z}_J, {\sum_{|J|=q}}' v_J\, d\bar{z}_J \Big) = {\sum_{|J|=q}}' \int_\Omega u_J(z) \overline{v_J(z)}\, dV(z) \tag{2.1}$$

We define $\bar\partial\colon\ \mathscr{L}^2_{(0,q)}(\Omega) \to \mathscr{L}^2_{(0,q+1)}(\Omega)$ as

$$\bar\partial \Big({\sum_{|J|=q}}' u_J\, d\bar{z}_J \Big) := \sum_{j=1}^{n} {\sum_{|J|=q}}' \frac{\partial u_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_J \tag{2.2}$$

with $\mathrm{dom}(\bar\partial_q) := \{u \in \mathscr{L}^2_{(0,q)}(\Omega) \mid \bar\partial u \in \mathscr{L}^2_{(0,q)}(\Omega)\}$. This gives rise to the $\bar\partial$- (or Dolbeault) complex

$$\mathscr{L}^2(\Omega) \xrightarrow{\bar\partial} \mathscr{L}^2_{(0,1)}(\Omega) \xrightarrow{\bar\partial} \mathscr{L}^2_{(0,2)}(\Omega) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{L}^2_{(0,n)}(\Omega) \xrightarrow{\bar\partial} 0. \tag{2.3}$$

(2.3) is indeed a complex, that is, $\bar\partial\bar\partial = 0$. This can be seen by direct computation. An alternative, and more illuminating, argument is obtained by also considering $\partial\colon \mathscr{L}^2_{(0,q)}(\Omega) \to \mathscr{L}^2_{(1,q)}(\Omega)$ defined by

$$\partial \Big({\sum_{|J|=q}}' u_J\, d\bar{z}_J \Big) = {\sum_{|J|=q}}' \sum_{j=1}^{n} \frac{\partial u_J}{\partial z_j} dz_j \wedge d\bar{z}_J. \tag{2.4}$$

Then $\partial + \bar\partial = d$, the de Rham exterior derivative. Hence

$$0 = d^2u = (\partial + \bar\partial)(\partial + \bar\partial)u = \partial\partial u + \partial\bar\partial u + \bar\partial\partial u + \bar\partial\bar\partial u.$$

The bidegrees of $\partial\partial u$, $\partial\bar\partial u + \bar\partial\partial u$, and $\bar\partial\bar\partial u$ are, respectively, $(2, q)$, $(1, q+1)$, $(0, q+2)$. If forms of different bidegrees add up to zero, each term has to be zero. Thus $\partial\partial = 0$, $\partial\bar\partial = -\bar\partial\partial$, and $\bar\partial\bar\partial = 0$.

Lemma 2.1. *For $0 \le q \le n$, $\bar\partial$ $(= \bar\partial_q)$ is a closed, densely defined operator.*

Proof. $\bar\partial$ is defined on all smooth compactly supported forms; these are dense in $\mathcal{L}^2_{(0,q)}(\Omega)$. To see that $\bar\partial$ is closed, observe that $u_n \to u$ in $\mathcal{L}^2_{(0,q)}(\Omega)$ implies that $\bar\partial u_n \to \bar\partial u$ as distributions. Therefore, if also $\bar\partial u_n \to v$ in $\mathcal{L}^2_{(0,q+1)}(\Omega)$, $u \in \mathrm{dom}(\bar\partial_q)$ and $\bar\partial u = v$. □

It follows from Lemma 2.1 that $\ker(\bar\partial_q) \subseteq \mathcal{L}^2_{(0,q)}(\Omega)$ is closed. Lemma 2.1 also implies that $(\bar\partial_q)$ has a Hilbert space adjoint

$$(\bar\partial_q)^*\colon\ \mathcal{L}^2_{(0,q+1)}(\Omega) \to \mathcal{L}^2_{(0,q)}(\Omega).$$

Recall that abstractly, $\bar\partial_q^*$ is given as follows. $v \in \mathcal{L}^2_{(0,q+1)}(\Omega) \in \mathrm{dom}(\bar\partial_q^*)$ if and only if there exists a constant such that $|(v, \bar\partial u)| \le C\|u\|$ for all $u \in \mathrm{dom}(\bar\partial_q)$. Then there exists $\hat v \in \mathcal{L}^2_{(0,q)}(\Omega)$ such that $(v, \bar\partial u) = (\hat v, u)$ for all $u \in \mathrm{dom}(\bar\partial_q)$. Since $\mathrm{dom}(\bar\partial_q)$ is dense in $\mathcal{L}^2_{(0,q)}(\Omega)$, $\hat v$ is unique, and we set $(\bar\partial_q)^* v := \hat v$.

General Hilbert space theory gives the following orthogonal decomposition:

$$\mathcal{L}^2_{(0,q)}(\Omega) = \ker(\bar\partial_q) \oplus \overline{\mathrm{Im}(\bar\partial_q^*)}. \tag{2.5}$$

Also, $\mathrm{Im}(\bar\partial_q^*) \subseteq \ker(\bar\partial_{q-1}^*)$ (if $q \ge 1$, because $\bar\partial\bar\partial = 0$); therefore $\overline{\mathrm{Im}(\bar\partial_q^*)} \subseteq \ker(\bar\partial_{q-1}^*)$. The orthogonal projection from $\mathcal{L}^2_{(0,q)}(\Omega)$ onto $\ker(\bar\partial_q)$ is called the Bergman projection; we denote it by P_q, $0 \le q \le n$:

$$P_q\colon \mathcal{L}^2_{(0,q)}(\Omega) \to \ker(\bar\partial_q). \tag{2.6}$$

We next compute $\bar\partial_q^*$. For this, assume Ω is bounded and smooth enough to allow integration by parts, say C^2. Let ρ be a defining function that satisfies $|\nabla\rho| = 1$ on $b\Omega$. Let $u \in C^1_{(0,q+1)}(\overline{\Omega})$, $\alpha \in C^\infty_{(0,q)}(\overline{\Omega})$. Then

$$\begin{aligned}(u, \bar\partial\alpha) &= \Big(\sideset{}{'}\sum_{|J|=q+1} u_J\, d\bar z_J, \sum_{j=1}^n \sideset{}{'}\sum_{|K|=q} \frac{\partial\alpha_K}{\partial\bar z_j}\, d\bar z_j \wedge d\bar z_K\Big) \\ &= \sum_{j=1}^n \sideset{}{'}\sum_{|K|=q} \int_\Omega u_{jK} \frac{\partial\overline{\alpha_K}}{\partial z_j}\, dV\end{aligned} \tag{2.7}$$

$$= -\sum_{j=1}^{n} \sum_{|K|=q}' \int_{\Omega} \frac{\partial u_{jK}}{\partial z_j} \overline{\alpha_K} \, dV + \sum_{j=1}^{n} \sum_{|K|=q}' \int_{b\Omega} u_{jK} \overline{\alpha_K} \frac{\partial \rho}{\partial z_j} \, d\sigma$$

$$= \Big(\sum_{|K|=q}' \Big(-\sum_{j=1}^{n} \frac{\partial u_{jK}}{\partial z_j} \Big) d\bar{z}_K, \sum_{|K|=q}' \alpha_K \, d\bar{z}_K \Big) + \sum_{|K|=q}' \int_{b\Omega} \Big(\sum_{j=1}^{n} u_{jK} \frac{\partial \rho}{\partial z_j} \Big) \overline{\alpha_K} \, d\sigma.$$

The same computation with a compactly supported α works for $u \in \mathcal{L}^2_{(0,q+1)}(\Omega)$ and without any boundary regularity assumptions on Ω. It results in the right-hand side of (2.7) without the boundary term. Therefore, if $u \in \operatorname{dom}(\bar{\partial}^*)$, then

$$\bar{\partial}^* u = -\sum_{|K|=q}' \Big(\sum_{j=1}^{n} \frac{\partial u_{jK}}{\partial z_j} \Big) d\bar{z}_K, \quad u \in \operatorname{dom}(\bar{\partial}^*). \tag{2.8}$$

The expression on the right-hand side of (2.8) can be computed for any u (if derivatives are taken as distributions); this is denoted by ϑu and is called the *formal* adjoint of $\bar{\partial}$. It is important to note that even if $\vartheta u \in \mathcal{L}^2_{(0,q)}(\Omega)$, u need not be in the domain of $\bar{\partial}^*$. This is also obvious from the boundary condition which we now compute. Returning to (2.7), we conclude that if $u \in C^1_{(0,q+1)}(\overline{\Omega}) \cap \operatorname{dom}(\bar{\partial}^*)$ (and Ω with at least C^2 boundary), then the boundary term on the right-hand side of (2.7) has to vanish for *all* $\alpha \in C^\infty_{(0,q)}(\overline{\Omega})$. Consequently,

$$\sum_{j=1}^{n} u_{jK} \frac{\partial \rho}{\partial z_j} = 0 \quad \text{on } b\Omega \text{ for all } K, \tag{2.9}$$

if $u \in C^1_{(0,q+1)}(\overline{\Omega}) \cap \operatorname{dom}(\bar{\partial}^*)$. In fact, the boundary condition (2.9) is necessary and sufficient for $u \in C^1_{(0,q+1)}(\overline{\Omega})$ to be in the domain of $\bar{\partial}^*$. To see that the condition is sufficient, note that the computation in (2.7) shows that $(u, \bar{\partial}\alpha) = (\vartheta u, \alpha)$ when (2.9) holds and $\alpha \in C^\infty_{(0,q)}(\overline{\Omega})$. In view of Proposition 2.3 below, $C^\infty_{(0,q)}(\overline{\Omega})$ is dense in $\operatorname{dom}(\bar{\partial})$ in the graph norm. Therefore, $(u, \bar{\partial}\alpha) = (\vartheta u, \alpha)$ for all $\alpha \in \operatorname{dom}(\bar{\partial})$, whence $u \in \operatorname{dom}(\bar{\partial}^*)$ (and $\bar{\partial}^* u = \vartheta u$).

It is useful to know that the domain of $\bar{\partial}^*$ is preserved under multiplication by a function in $C^1(\overline{\Omega})$. Let $u \in \operatorname{dom}(\bar{\partial}^*)$, $v \in \operatorname{dom}(\bar{\partial})$, and $\psi \in C^1(\overline{\Omega})$. Then

$$(\bar{\partial} v, \psi u) = (\bar{\psi} \bar{\partial} v, u) = (\bar{\partial}(\bar{\psi} v), u) - (\bar{\partial}\bar{\psi} \wedge v, u) = (\bar{\psi} v, \bar{\partial}^* u) - (\bar{\partial}\bar{\psi} \wedge v, u). \tag{2.10}$$

The right-hand side of (2.10) is bounded by $\|v\|$; therefore, $\psi u \in \mathrm{dom}(\bar{\partial}^*)$. Then

$$\bar{\partial}^*(\psi u) = \vartheta(\psi u) = \psi\bar{\partial}^* u - \sum_K{}' \Big(\sum_j (\partial\psi/\partial z_j) u_{jK}\Big) d\bar{z}_K. \tag{2.11}$$

Alternatively, (2.11) results directly from (2.10).

2.2 Special boundary charts

We now introduce some special vector fields and (1,0)-forms associated with $b\Omega$. Let Ω have boundary of class at least C^2. For $P \in b\Omega$, the complex tangent space $T_P^{\mathbb{C}}(b\Omega)$ is given by those $w \in \mathbb{C}^n$ satisfying $\sum_{j=1}^n (\partial\rho/\partial z_j)(P) w_j = 0$, where ρ is a defining function for Ω. It is easy to see that this definition is independent of the defining function chosen. $T_P^{\mathbb{C}}(b\Omega)$ is a complex subspace of $\mathbb{C}^n$; it is the maximal complex subspace of $T_P(b\Omega)$. Near a point $P \in b\Omega$, choose fields $L_1, L_2, \ldots, L_{n-1}$ of type (1,0) that are orthonormal and that span $T_z^{\mathbb{C}}(b\Omega_\varepsilon)$ (z near P), where $\Omega_\varepsilon = \{z \in \Omega | \rho(z) < -\varepsilon\}$. This can be done by first just choosing a basis, and then using the Gram–Schmidt process. To this collection add L_n, the complex normal, normalized to have length 1. (So L_n is a smooth multiple of $\sum_j \frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial}{\partial z_j}$.) Note that in contrast to $L_1, L_2, \ldots, L_{n-1}$, L_n is globally defined. Now denote by $\omega_1, \omega_2, \ldots, \omega_n$ the (1,0)-forms dual to $L_1, \ldots, L_n$, i.e., $\omega_j(L_k) = \delta_{jk}$. The ω_j's then form an orthonormal basis for the (1,0)-forms near P. ω_n is a smooth multiple of $\sum_{j=1}^n \frac{\partial\rho}{\partial z_j} dz_j$, and is again defined globally. Taking wedge products of the ω_j's yields (local) orthonormal bases for $(q,0)$-forms also when $q > 1$. $\{\omega_1, \ldots, \omega_n\}$ is called a *special boundary frame*. We can also choose coordinates $(t_1, t_2, \ldots, t_{2n-1}, \rho)$ near $P \in b\Omega$ such that ρ is a defining function for Ω and $(t_1, \ldots, t_{2n-1}, 0)$ are coordinates on $b\Omega$ (near P). These coordinates together with a special boundary frame are referred to as a *special boundary chart* in [125], where the technique is ascribed to [4].

One of the reasons that special boundary frames are very useful is that the boundary condition (2.9) takes on an especially simple form when expressed in a special boundary frame. Let $L_j = \sum_{s=1}^n a_{js}\frac{\partial}{\partial z_s}$, $\omega_j = \sum_{s=1}^n b_{js} dz_s$, $1 \le j \le n$. Then

$$\delta_{jk} = \omega_j(L_k) = \sum_{s=1}^n b_{js} dz_s\Big(\sum_{m=1}^n a_{km}\frac{\partial}{\partial z_m}\Big) = \sum_{s=1}^n b_{js} a_{ks}. \tag{2.12}$$

Consequently, if f is a function,

$$\bar{\partial} f = \sum_{s=1}^n \frac{\partial f}{\partial\bar{z}_s} d\bar{z}_s = \sum_{jks} \overline{a^{sk}}(\overline{L}_k f)\overline{b^{sj}}\,\overline{\omega}_j = \sum_{j=1}^n (\overline{L}_j f)\overline{\omega}_j, \tag{2.13}$$

where the superscripts denote the (entries of the) inverses of the corresponding matrices with subscripts. Since multiplication by functions in $C^\infty(\overline{\Omega})$ preserves $\mathrm{dom}(\bar{\partial}^*)$, we may assume that the form u is supported in a special boundary chart. So $u = \sum'_J u_J\bar{\omega}_J$, where $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_q}$, with $u_J \in C^1(\bar{\Omega})$ (i.e., $u \in C^1_{(0,q)}(\bar{\Omega})$). Then, in view of (2.13)

$$\begin{aligned}\bar{\partial}u &= \bar{\partial}\Big({\sum_J}' u_J\bar{\omega}_J\Big)\\ &= {\sum_J}'\Big(\bar{\partial}u_J \wedge \bar{\omega}_J + u_J\bar{\partial}\bar{\omega}_J\Big)\\ &= {\sum_J}'\sum_{j=1}^{n}(\overline{L_j}u_J)\bar{\omega}_j \wedge \bar{\omega}_J + {\sum_J}' u_J\bar{\partial}\bar{\omega}_J.\end{aligned} \tag{2.14}$$

Arguing now as in (2.7), (2.8) above, but computing the inner product $(u, \bar{\partial}\alpha)$ in terms of the coefficients with respect to the basis $\{\bar{\omega}_J | J \text{ increasing}, |J| = q\}$ reveals that

$$u \in \mathrm{dom}(\bar{\partial}^*) \iff u_J = 0 \text{ on } b\Omega \text{ when } n \in J. \tag{2.15}$$

Indeed, the only boundary terms that arise come from integrating $\int_\Omega \overline{L_n}\alpha_K\overline{u_{nK}}$ by parts and they equal $\int_{b\Omega} \alpha_K\overline{u_{nK}}(\overline{L_n}\rho)$. Since α_K can be arbitrary on $b\Omega$ (and $\overline{L_n}\rho \neq 0$ on $b\Omega$), we conclude $u_{nK} = 0$ on $b\Omega$ for all K. That the condition is sufficient follows as for (2.9). When $q = 1$, (2.15) simply says that $u_n = 0$ on $b\Omega$: the 'normal component of u' vanishes on the boundary. Alternatively, (2.15) can be derived from (2.9) directly, either via basis change computations analogous to (2.12), (2.13), or by expressing (2.9) in a basis free manner. Namely, the left-hand side of (2.9) equals the K-th coefficient of the interior product of $\overline{L_n}$ with u. Thus this interior product should vanish on the boundary. Expressing this in a special boundary frame gives (2.15).

These arguments also give the formula for ϑ and $\bar{\partial}^*$, respectively, in special boundary frames:

$$\vartheta u = \vartheta\Big({\sum_J}' u_J\bar{\omega}_J\Big) = -{\sum_{|K|=q-1}}'\Big(\sum_{j=1}^{n} L_j u_{jK}\Big)\bar{\omega}_K + \text{0-th order}(u). \tag{2.16}$$

0-th order(u) indicates terms that contain no derivatives of the u_J's. When $u \in \mathrm{dom}(\bar{\partial}^*)$, (2.16) gives $\bar{\partial}^* u$.

2.3 Sobolev spaces of forms

We will also consider ($\mathcal{L}^2$) Sobolev spaces of forms. We use the usual notation $W^s(\Omega)$, $s \in \mathbb{R}$. $W^s_0(\Omega)$ denotes the closure of the smooth compactly supported functions in $W^s(\Omega)$. Detailed information on Sobolev spaces may be found for example in [215], [299], [294]. When needed, we will use a subscript s, occasionally also a subscript $W^s(\Omega)$, to denote the norm in $W^s(\Omega)$. The Sobolev spaces of forms are defined coefficientwise, i.e., a form $u = \sum'_J u_J d\bar{z}_J$ is in $W^s(\Omega)$ if and only if $u_J \in W^s(\Omega)$ for all J. We use a subscript $(0, q)$ to indicate spaces of forms, i.e., $W^s_{(0,q)}(\Omega)$, $W^s_{0,(0,q)}(\Omega)$, etc. This convention is also adopted for forms with coefficients in other function spaces, such as $C^\infty_{(0,q)}(\overline{\Omega})$, $C^k_{(0,q)}(\overline{\Omega})$, etc. The reader should note that when we change to special boundary frames, the coefficients in general need not be in the same space, unless the boundary is regular enough. When k derivatives are involved, the boundary should be at least C^{k+1}: then the normal is C^k, which implies that the coefficients in the formulas passing from the Euclidean frame to a special boundary frame, or vice versa, are in $C^k(\overline{\Omega})$.

It will be seen in subsequent chapters that it is important to be able to work with 'derivatives', i.e., first order partial differential operators or vector fields, which preserve the domain of $\bar{\partial}^*$. Assume that the boundary is regular enough that when computing a certain Sobolev norm of a form u, we may compute $\mathcal{L}^2$-norms of derivatives of the coefficients of u in a collection of boundary frames. Given (2.15) for membership in $\mathrm{dom}(\bar{\partial}^*)$, it is natural to consider tangential vector fields and let them act in special boundary frames. Then the right-hand side of (2.15) is preserved. However, this breaks down for the normal derivative. The next lemma takes care of this problem: normal derivatives of u are controlled by tangential ones, $\bar{\partial}u$, $\bar{\partial}^*u$, and u itself, and $u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ is in $W^1_{(0,q)}(\Omega)$ as soon as all tangential derivatives are square integrable.

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^{k+1} boundary, $k \geq 1$. Denote by $\partial/\partial\nu$ the normal derivative, near $b\Omega$. Then, if u is a $(0, q)$-form, $(\partial u/\partial\nu)$ (computed coefficientwise) can be expressed as follows:*

$$\frac{\partial u}{\partial \nu} = \begin{array}{l}\textit{linear combination, with coefficients in } C^k(\overline{\Omega}),\\ \textit{of coefficients of } \bar{\partial}u,\ \vartheta u,\ u, \textit{ and tangential derivatives of } u.\end{array} \tag{2.17}$$

We remark that in somewhat more sophisticated language, (2.17) is an expression of the fact that $b\Omega$ is not characteristic for the complex $\bar{\partial} \oplus \vartheta$. We have not specified the regularity of u: we interpret (2.17) as equality between first order partial differential operators (on $(0, q)$-forms) with $C^k(\overline{\Omega})$ coefficients. For this, it suffices to check equality on $u \in C^1_{(0,q)}(\overline{\Omega})$. Note that if (2.17) holds when the differential operators involved act componentwise in some frame, then it holds when the operators act componentwise in a different frame. This results from the transformation formulas from one frame to another (i.e., linear combinations with coefficients in $C^k(\overline{\Omega})$).

Proof of Lemma 2.2. Via a suitable partition of unity on $\overline{\Omega}$, it suffices to check (2.17) in a special boundary frame. Since $\partial/\partial\nu = (1/2)(L_n + \overline{L_n})$, and $L_n - \overline{L_n}$ is tangential, it suffices to see that for each coefficient u_J of u, $L_n u_J$ or $\overline{L_n} u_J$ has the required form. When $n \in J$, we use (2.16) and read off that $L_n u_{nK}$ is of the form required in (2.17). When $n \notin J$, (2.14) shows that $\overline{L_n} u_J$ is of the form needed. □

2.4 Density of 'nice' forms

'Nice' forms are dense in the graph norm of $\bar\partial^*$, $\bar\partial$, and $\bar\partial \oplus \bar\partial^*$, respectively. (ii) and (iii) below are in Hörmander ([170]), a version of (i) is pointed out in [81].

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$.*

(i) $C^\infty_{0,(0,q)}(\Omega)$ *is dense in* $\mathrm{dom}(\bar\partial^*)$ *in the graph norm* $u \mapsto (\|u\|^2 + \|\bar\partial^* u\|^2)^{1/2}$.

(ii) *If $b\Omega$ is C^1, then* $C^\infty_{(0,q)}(\overline{\Omega})$ *is dense in* $\mathrm{dom}(\bar\partial)$ *in the graph norm* $u \mapsto (\|u\|^2 + \|\bar\partial u\|^2)^{1/2}$.

(iii) *Let $k \geq 1$. If $b\Omega$ is C^{k+1}, then* $C^k_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar\partial^*)$ *is dense in* $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$ *in the graph norm* $u \mapsto (\|u\|^2 + \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2)^{1/2}$. *The statement also holds with $k+1$ and k replaced by ∞.*

The reader should note that compactly supported forms are not dense in $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$ (in the graph norm). For compactly supported forms, (the computation in the proof of) Proposition 2.4 below gives

$$\sum_{|J|=q}{}' \sum_{j=1}^{n} \left\| \frac{\partial u_J}{\partial \bar z_j} \right\|^2 = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \tag{2.18}$$

for $u = \sum'_{|J|=q} u_J \, d\bar z_J$, with u_J compactly supported. Integration by parts also shows that in this case $\left\|\partial u_J/\partial z_j\right\|^2 = \left\|\partial u_J/\partial \bar z_j\right\|^2$, so that

$$\|u\|_1^2 \leq 2(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2), \quad u \in C^\infty_{0,(0,q)}(\Omega), \tag{2.19}$$

where $\|u\|_1$ denotes the standard Sobolev-1 norm of u on Ω. In particular, the closure of the compactly supported forms in $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$ in the graph norm is contained in the Sobolev space $W^1_0(\Omega)$. For forms that are (for example) C^1 on $\overline{\Omega}$, this means that they are zero on the boundary. This is considerably stronger than (2.9).

Proof of Proposition 2.3. In [170] (see Proposition 2.1.1), the result is obtained as a special case of results about more general systems of differential equations (compare [209]; see also the exposition in [212], Chapter V). A proof for the case of $\bar\partial \oplus \bar\partial^*$ is in[81], Lemma 4.3.2. We combine elements from both [170] and [81].

When Ω satisfies some boundary regularity (for example C^1), part (i) can be proved using mollifiers carefully (see [81], proof of Lemma 4.3.2). When no boundary regularity is assumed, a soft (duality) argument still gives the result. It suffices to see that if $u \in \mathrm{dom}(\bar\partial^*)$ is orthogonal to all smooth compactly supported forms (in the inner product corresponding to the graph norm), then $u = 0$. So assume $(\bar\partial^* u, \bar\partial^* v) + (u, v) = 0$ for all smooth compactly supported v. The first inner product equals $(\bar\partial\bar\partial^* u, v)$, where the first term is computed as a distribution, and the pairing is now between a distribution and a test function (form). Therefore, we obtain $(\bar\partial\bar\partial^* u + u, v) = 0$ for all such v, whence $\bar\partial\bar\partial^* u + u = 0$. But u is in $\mathcal{L}^2$, therefore $\bar\partial\bar\partial^* u$ is in $\mathcal{L}^2$, i.e., $\bar\partial^* u \in \mathrm{dom}(\bar\partial)$. Then $0 = (\bar\partial\bar\partial^* u, u) + (u, u) = \|\bar\partial^* u\|^2 + \|u\|^2$. Thus $u = 0$, as required.

To see (ii), note that we may assume that the form u is supported near a boundary point, via a suitable partition of unity on $\bar\Omega$: the compactly supported part can be approximated (in the graph norm!) by convolution with standard mollifiers. For u supported near a boundary point P, we choose the mollifiers slightly more carefully, in such a way that 'jumps' across the boundary do not interfere. Pick a (small) neighborhood U of P, an open cone Γ (with vertex at the origin of $\mathbb{C}^n$), and $a > 0$ so that $z - \zeta \in \Omega$ for all $z \in U, \zeta \in \Gamma, |\zeta| < a$. This can be done because $b\Omega$ is C^1. We may assume that u is compactly supported in $U \cap \bar\Omega$. Choose $\varphi \in C_0^\infty(\Gamma \cap B(0, a))$, with $\varphi \geq 0$, and $\int \varphi = 1$. Here, $B(0, a)$ denotes the open ball of radius a, centered at 0. For $\varepsilon > 0$, set $\varphi_\varepsilon(z) = \varepsilon^{-2n}\varphi(\frac{z}{\varepsilon})$. Denote by a superscript tilde the extension to $\mathbb{C}^n$ of a function/form in $\mathcal{L}^2(U \cap \bar\Omega)$ by setting it zero outside $U \cap \bar\Omega$. Finally, use $*$ to denote convolution. Then $u_\varepsilon := \varphi_\varepsilon * \tilde u \in C^\infty_{(0,q)}(\bar\Omega)$ (as the restriction of a form in $C^\infty(\mathbb{C}^n)$). Moreover, $\bar\partial u_\varepsilon = \varphi_\varepsilon * \bar\partial\tilde u$, so that on Ω, $\bar\partial u_\varepsilon = \varphi_\varepsilon * \left(\widetilde{\bar\partial u}\right)$ (again because φ, hence φ_ε, is supported in $\Gamma \cap B(0, a)$). Consequently, $u_\varepsilon \to u$ in the graph norm as $\varepsilon \to 0$.

Part (iii) is more subtle, because the approximating forms need to be in $\mathrm{dom}(\bar\partial^*)$, that is, their normal component must vanish on the boundary. To achieve this, it is again convenient to work in a special boundary chart. So assume $u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$ is supported in a special boundary chart U, $u = \sum'_J u_J \bar\omega_J$, and u is compactly supported in $U \cap \bar\Omega$ (note again that multiplication by a smooth function preserves $\mathrm{dom}(\bar\partial^*)$). We may assume that $(t_1, \dots, t_{2n-1}, r)$ are coordinates in U such that r is plus or minus the boundary distance, depending on whether the point is outside or inside Ω. Then $\partial/\partial r$ is the unit vector field perpendicular to the level sets of r, and $L_n = (1/\sqrt{2})(\partial/\partial r + iT)$, where $T = \sqrt{2}\,\mathrm{Im}\, L_n$ is a (real) unit vector field tangential to the level sets of r and perpendicular to the complex tangent space of these level sets.

Choose $\psi \in C_0^\infty(\mathbb{R}^{2n-1})$ with $\int_{\mathbb{R}^{2n-1}} \psi = 1$ and $\psi(-x) = \psi(x)$. Set $\psi_\varepsilon(x) = (1/\varepsilon^{2n-1})\psi(x/\varepsilon)$. Define the measure μ_ε with support in $\{r = 0\}$ by $\int_{\mathbb{R}^{2n}} f\, d\mu_\varepsilon = \int_{\mathbb{R}^{2n-1}} f(t_1, \dots, t_{2n-1}, 0)\ \psi_\varepsilon(t_1, \dots, t_{2n-1}) dV(t_1, \dots, t_{2n-1})$ for $f \in C(\mathbb{R}^{2n})$. Set $u_\varepsilon = \sum' u_{J,\varepsilon}\bar\omega_J$, where $u_{J,\varepsilon} = \mu_\varepsilon * u_J$. That is, we have smoothed u only in tangential directions. Intuitively, this should preserve the domain of $\bar\partial^*$. While this is obvious for $u \in C^1_{(0,q)}(\bar\Omega)$, the case of u only in $\mathrm{dom}(\bar\partial) \cap$ $\mathrm{dom}(\bar\partial^*)$ requires some care. First note that by standard facts about convolutions, $u_\varepsilon \in \mathcal{L}^2_{(0,q)}(\Omega)$ and $u_\varepsilon \to u$ in $\mathcal{L}^2_{(0,q)}(\Omega)$. Next, observe that because of part (ii) of the proposition already proved, it suffices to pair u_ε with $\bar\partial v$ for $v \in C^\infty_{(0,q-1)}(\bar\Omega)$. In the special boundary chart, $v \in C^1(\mathbb{R}^{2n}_-)$ on the support of u, where $\mathbb{R}^{2n}_- = \{r < 0\}$. We can now repeat the discussion in the proof of (2.15). Because $u_{J,\varepsilon}(t_1, \dots, t_{2n-1}, r)$ is smooth for almost all $r > 0$, we can integrate the tangential operators $\bar L_j$, $1 \leq j \leq n-1$ by parts without boundary terms. The result is in $\mathcal{L}^2(\mathbb{R}^{2n}_-)$: the coefficients of these operators in the special boundary chart, as well as the Jacobian of the coordinate transformation, are in $C^k(\mathbb{R}^{2n}_-)$. Similarly, the tangential operator T in $\overline{L_n} = 1/\sqrt{2}(\partial/\partial r + iT)$ can be integrated by parts without boundary term. Computing the inner product $(\bar{\partial}v, u_\varepsilon)$ in the special boundary chart, we therefore obtain, *modulo terms that are* $O_{\varepsilon,u}(\|v\|)$,

$$
\begin{aligned}
(\bar{\partial}v, u_\varepsilon) &\approx \sum_{|K|=q-1}{}' (\sqrt{2})^{-1} \int_{\mathbb{R}^{2n}_-} \frac{\partial v_K}{\partial r} \overline{\mu_\varepsilon * u_{nK}} \\
&\approx \sum_{|K|=q-1}{}' (\sqrt{2})^{-1} \int_{\mathbb{R}^{2n}_-} \frac{\partial(\mu_\varepsilon * v_K)}{\partial r} \overline{u_{nK}} \\
&\approx (\bar{\partial}(\mu_\varepsilon * v), u) = (\mu_\varepsilon * v, \bar{\partial}^* u).
\end{aligned} \tag{2.20}
$$

We have used here that convolution with μ_ε is self-adjoint in $\mathcal{L}^2(\mathbb{R}^{2n}_-)$ (by our choice of the mollifier ψ above), and that it commutes with $\partial/\partial r$. In the last step in (2.20), note that v is smooth up to the boundary on Ω and therefore, $\mu_\varepsilon * v$ is C^k up to the boundary of $\mathbb{R}^{2n}_-$ (on the support of u). In particular, $\mu_\varepsilon * v \in \mathrm{dom}(\bar{\partial})$. (2.20) shows that $|(\bar{\partial}v, u_\varepsilon)| \le C_{\varepsilon,u}\|v\|$, that is, $u_\varepsilon \in \mathrm{dom}(\bar{\partial}^*)$.

Friedrichs' Lemma about commutators between convolutions with families such as $\{\psi_\varepsilon\}_{\varepsilon>0}$ and first order partial differential operators says that the commutators tend to zero in $\mathcal{L}^2(\mathbb{R}^{2n-1})$ when ε tends to zero (see for example [299], Lemma 25.4 or [81], Lemma D.1). Applying this lemma in the variables $(t_1, \dots, t_{2n-1})$ for $r < 0$ fixed to the convolutions with μ_ε (i.e., convolutions with ψ_ε), together with the fact that $\partial/\partial r$ commutes with these convolutions, shows firstly that $\bar{\partial}u_\varepsilon \in \mathcal{L}^2$ (i.e., $u_\varepsilon \in \mathrm{dom}(\bar{\partial})$), and secondly that $\mu_\varepsilon * (\bar{\partial}u) - \bar{\partial}u_\varepsilon \to 0$ and $\mu_\varepsilon * (\bar{\partial}^* u) - \bar{\partial}^* u_\varepsilon \to 0$ in the respective $\mathcal{L}^2$ spaces. Thus $u_\varepsilon \to u$ in the graph norm of $\bar{\partial} \oplus \bar{\partial}^*$.

Since the tangential derivatives of u_ε are in $\mathcal{L}^2$, we infer from Lemma 2.2 that $u_\varepsilon \in W^1(\Omega)$. Approximating u_ε in $W^1(\Omega)$ by forms with coefficients in $C^\infty(\overline{\Omega})$ shows that (2.7) holds when $u_\varepsilon|_{b\Omega}$ is interpreted in the sense of a trace in $\mathcal{L}^2(b\Omega)$ (the relevant trace and density theorems may be found, for example, in[122], Chapter 5 ; density of $C^\infty(\overline{\Omega})$ in $W^1(\Omega)$ also follows from the argument in (ii)). Arguing now as in (2.9) (or in (2.15)) shows that the functions $(u_\varepsilon)_{nK}$ have trace zero on $b\Omega$ for all multi-indices K of length $q-1$, and consequently can be approximated, in $W^1(\Omega)$, by smooth compactly supported functions (see again [122]). Approximating the functions $(u_\varepsilon)_J$ with $n \notin J$ in $W^1(\Omega)$ by functions in $C^\infty(\overline{\Omega})$ yields approximation in $W^1(\Omega)$ of u_ε by forms in $C^k(\overline{\Omega})$ with compactly supported normal components. (The approximating forms have coefficients in $C^\infty(\overline{\Omega})$ in the special boundary frame, but in general only in $C^k(\overline{\Omega})$ in the Euclidean frame, since $b\Omega$ is only assumed C^{k+1}.) Such forms are in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$, and the approximation is in the graph norm (since it is in $W^1(\Omega)$).

When k and $k+1$ are infinity, the above arguments prove density of forms in $C^\infty(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*)$. This concludes the proof of Proposition 2.3. □

2.5 Weighted norms

It will be useful to consider weighted $\mathcal{L}^2$-norms as well. The weight will be of the form $e^{-\varphi}$ for suitable functions φ; we denote these norms by $\|\cdot\|_\varphi$. The $\bar\partial$-complex can be set up with respect to weighted $\mathcal{L}^2$-spaces as well; we will denote the resulting adjoint by $\bar\partial^*_\varphi$ and the resulting formal adjoint by ϑ_φ. Note that ϑ and ϑ_φ are related as follows:

$$\vartheta_\varphi u = \vartheta u + \sum_{j=1}^{n} {\sum_{|K|=q}}' \frac{\partial\varphi}{\partial z_j} u_{jK}\, d\bar z_K, \tag{2.21}$$

where $u = \sum'_{|J|=q+1} u_J\, d\bar z_J$. This can be seen by pairing with $\bar\partial$ of a compactly supported form (as in (2.21)). In particular, ϑ_φ and ϑ differ by an operator of order zero (i.e., involving no derivatives). More generally, for $u \in \mathrm{dom}(\bar\partial^*)$ and $\varphi \in C^1(\bar\Omega)$

$$\begin{aligned}(u, \bar\partial v)_\varphi &= (u, e^{-\varphi}\bar\partial v) = (u, \bar\partial(e^{-\varphi}v)) - (u, \bar\partial(e^{-\varphi}) \wedge v),\\ &= (\bar\partial^* u, e^{-\varphi}v) - (u, \bar\partial(e^{-\varphi}) \wedge v)\\ &= (\bar\partial^* u, v)_\varphi + (u, \bar\partial\varphi \wedge v)_\varphi.\end{aligned} \tag{2.22}$$

The right-hand side of (2.22) is bounded by $C(u)\|v\|_q$, so that $u \in \mathrm{dom}(\bar\partial^*_\varphi)$ (by definition of $\bar\partial^*_\varphi$). Similarly; $\mathrm{dom}(\bar\partial^*_\varphi) \subseteq \mathrm{dom}(\bar\partial^*)$. Consequently, $\mathrm{dom}(\bar\partial^*_\varphi) = \mathrm{dom}(\bar\partial^*)$; moreover, (2.21) holds with ϑ_φ and ϑ replaced by $\bar\partial^*_\varphi$ and $\bar\partial^*$, respectively:

$$\bar\partial^*_\varphi u = \bar\partial^* u + \sum_{j=1}^{n} {\sum_{|K|=q}}' \frac{\partial\varphi}{\partial z_j} u_{jK}\, d\bar z_K, \quad u \in \mathrm{dom}(\bar\partial^*_\varphi) = \mathrm{dom}(\bar\partial^*). \tag{2.23}$$

Remark. Proposition 2.3 continues to hold for the weighted operators (when $\varphi \in C^1(\bar\Omega)$). Indeed, in view of (2.23), the same approximating sequences will work (since the $\mathcal{L}^2$ norms are comparable).

2.6 The 'twisted' Kohn–Morrey–Hörmander formula

The following identity, often referred to as the twisted Kohn–Morrey–Hörmander formula, is the key to the $\mathcal{L}^2$-theory.

Proposition 2.4. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary and (C^2) defining function ρ; let u be a $(0,q)$-form $(1 \le q \le n)$ that is in the domain of $(\bar\partial^*)$ and that is continuously differentiable on $\bar\Omega$; and let a and φ be real-valued functions that are twice continuously differentiable on $\bar\Omega$, with $a \ge 0$. Then*

$$\begin{aligned}\|\sqrt{a}\,\bar\partial u\|^2_\varphi + \|\sqrt{a}\,\bar\partial^*_\varphi u\|^2_\varphi &= {\sum_K}' \sum_{j,k=1}^{n} \int_{b\Omega} a \frac{\partial^2\rho}{\partial z_j \partial\bar z_k} u_{jK}\overline{u_{kK}} e^{-\varphi} \frac{d\sigma}{|\nabla\rho|}\\ &\quad + {\sum_J}' \sum_{j=1}^{n} \int_\Omega a \left|\frac{\partial u_J}{\partial\bar z_j}\right|^2 e^{-\varphi}\, dV\end{aligned} \tag{2.24}$$

$$+ 2\,\mathrm{Re}\left({\sum_K}' \sum_{j=1}^{n} u_{jK} \frac{\partial a}{\partial z_j} d\bar{z}_K, \bar{\partial}_\varphi^* u \right)_\varphi$$
$$+ {\sum_K}' \sum_{j,k=1}^{n} \int_\Omega \left(a \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} - \frac{\partial^2 a}{\partial z_j \partial \bar{z}_k} \right) u_{jK} \overline{u_{kK}} e^{-\varphi}\, dV.$$

Note that the defining function only enters into the boundary term on the right-hand side of (2.24). Computation of the complex Hessian of $\rho = e^g \tilde{\rho}$ (near $b\Omega$), and then using the boundary condition (2.9) for membership in the domain of $\bar{\partial}^*$ or of $\bar{\partial}_\varphi^*$ shows that this boundary term is indeed independent of the defining function. The case $a \equiv 1$ and $\varphi \equiv 0$ gives the classical Kohn–Morrey formula (see [188], [190], [228]; also [4]); the case $a \equiv 1$ (but φ non-constant) is in [170]. The idea of using an additional auxiliary function goes back to [244], [240]; it was further developed and put into the above form in [275], [220] (see also [243]). For a derivation of (2.24) different from the one below, the reader should consult [35]; useful additional perspective can be found in [36] and in [223].

Proof of Proposition 2.4. The proof uses integration by parts repeatedly; boundary terms are again handled using the boundary condition (2.9). When integrating by parts, it is convenient to have a normalized defining function ρ, i.e., a defining function ρ with $|\nabla \rho| = 1$ on $b\Omega$. If $b\Omega$ is C^k with $k \geq 2$, a normalized defining function which is also C^k may be obtained starting, near $b\Omega$, with the signed boundary distance (see e.g. [148], Lemma 14.16 for a proof that this boundary distance is C^k near $b\Omega$). By what was said above, it suffices to prove the proposition for such a defining function. The reader new to dealing with differential forms is advised to go through the proof first with $q = 1$ in mind; there are no new ideas needed when $q > 1$. Our presentation (which is by now standard) follows [170], [221].

Let $u = \sum'_{|J|=q} u_J\, d\bar{z}_J$ as in the proposition. Then

$$\bar{\partial} u = {\sum_{|J|=q}}' \sum_{j=1}^{n} \frac{\partial u_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_J \tag{2.25}$$

and

$$\bar{\partial}_\varphi^* u = - {\sum_{|K|=q-1}}' \sum_{k=1}^{n} \left(\frac{\partial}{\partial z_k} - \frac{\partial \varphi}{\partial z_k} \right) u_{kK}\, d\bar{z}_K \tag{2.26}$$

(compare the computation in (2.22)). It will be convenient to set

$$\begin{aligned} \delta_k &:= \frac{\partial}{\partial z_k} - \frac{\partial \varphi}{\partial z_k} \\ &= e^\varphi \frac{\partial}{\partial z_k}(e^{-\varphi}). \end{aligned} \tag{2.27}$$

We get

$$\|\sqrt{a}\,\bar{\partial}u\|_{\varphi}^{2}+\|\sqrt{a}\,\bar{\partial}_{\varphi}^{*}u\|_{\varphi}^{2}=\sum_{|J|=|M|=q}{}^{'}\sum_{j,k=1}^{n}\varepsilon_{jJ}^{kM}\int_{\Omega}a\frac{\partial u_J}{\partial\bar{z}_j}\overline{\frac{\partial u_M}{\partial\bar{z}_k}}e^{-\varphi}\,dV \\ +\sum_{|K|=q-1}{}^{'}\sum_{j,k}\int_{\Omega}a\delta_j u_{jK}\overline{\delta_k u_{kK}}e^{-\varphi}\,dV, \tag{2.28}$$

where $\varepsilon_{jJ}^{kM}=0$ if $j\in J$ or $k\in M$ or if $\{k\}\cup M\neq\{j\}\cup J$, and equals the sign of the permutation $\binom{kM}{jJ}$ otherwise. The right-hand side of (2.28) can be rewritten as

$$\sum_{|J|=q}{}^{'}\sum_{j=1}^{n}\left\|\sqrt{a}\,\frac{\partial u_J}{\partial\bar{z}_j}\right\|_{\varphi}^{2}+\sum_{|K|=q-1}{}^{'}\sum_{j,k=1}^{n}\int_{\Omega}a\left\{\delta_j u_{jK}\overline{\delta_k u_{kK}}-\frac{\partial u_{jK}}{\partial\bar{z}_k}\overline{\frac{\partial u_{kK}}{\partial\bar{z}_j}}\right\}e^{-\varphi}\,dV. \tag{2.29}$$

The second sum in (2.28) appears in (2.29) unchanged, and we only have to work on the first sum. Consider first the (nonzero) terms where $j=k$ (and hence $M=J$). These terms result in the portion of the first sum in (2.29) where $j\notin J$. On the other hand, when $j\neq k$, then $j\in M$ and $k\in J$, and deletion of j from M and k from J results in the same strictly increasing multi-index K of length $q-1$. Consequently, these terms can be collected into the second sum in (2.29) (the part with the minus sign, we have also interchanged the summation indices j and k). In this sum, the terms where $j=k$ compensate for the terms in the first sum where $j\in J$. Next, in the second sum in (2.29), we move the operators δ_k and $\partial/\partial\bar{z}_j$ to the left as $\partial/\partial\bar{z}_k$ and δ_j, respectively, by integration by parts. Note that

$$\left(\frac{\partial f}{\partial\bar{z}_k},g\right)_{\varphi}=-(f,\delta_k g)_{\varphi}+\int_{b\Omega}f\bar{g}\frac{\partial\rho}{\partial\bar{z}_k}e^{-\varphi}\,d\sigma, \tag{2.30}$$

for f and g sufficiently smooth. With this, we obtain that the second sum in (2.29) equals

$$\sum_{|K|=q-1}{}^{'}\sum_{j,k=1}^{n}\left\{\int_{\Omega}\left(-\frac{\partial}{\partial\bar{z}_k}(a\delta_j u_{jK})+\delta_j\left(a\frac{\partial u_{jK}}{\partial\bar{z}_k}\right)\right)\overline{u_{kK}}e^{-\varphi}\,dV\right. \\ \left.+\int_{b\Omega}a\delta_j u_{jK}\overline{u_{kK}}\frac{\partial\rho}{\partial\bar{z}_k}e^{-\varphi}\,d\sigma-\int_{b\Omega}a\frac{\partial u_{jK}}{\partial\bar{z}_k}\overline{u_{kK}}\frac{\partial\rho}{\partial z_j}e^{-\varphi}\,d\sigma\right\}. \tag{2.31}$$

The first boundary integral vanishes because $\sum_k\overline{u_{kK}}\frac{\partial\rho}{\partial\bar{z}_k}=\overline{\sum_k u_{kK}\frac{\partial\rho}{\partial z_k}}=0$ (since $u\in\mathrm{dom}(\bar{\partial}^{*})$). In the second boundary integral, the derivative $\sum_k\overline{u_{kK}}\frac{\partial}{\partial\bar{z}_k}$ is tangential (at boundary points). Integrating it by parts and observing again that $\sum_j u_{jK}\frac{\partial\rho}{\partial z_j}=0$ on the boundary gives that this term equals

$$\sum_{|K|=q-1}{}^{'}\sum_{j,k}\int_{b\Omega}a\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}u_{jK}\overline{u_{kK}}e^{-\varphi}\,d\sigma. \tag{2.32}$$

The first sum in (2.31) can be written as

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \left\{ \int_\Omega a \left[\delta_j, \frac{\partial}{\partial \bar{z}_k}\right] u_{jK}\overline{u_{kK}} e^{-\varphi}\, dV \right. \\ \left. + \int_\Omega \left(\frac{\partial a}{\partial z_j} \frac{\partial u_{jK}}{\partial \bar{z}_k} - \frac{\partial a}{\partial \bar{z}_k} \delta_j u_{jK} \right) \overline{u_{kK}} e^{-\varphi}\, dV \right\}. \tag{2.33}$$

Here $\left[\delta_j, \partial/\partial\bar{z}_k\right]$ denotes as usual the commutator between (differential) operators. It is easily computed to be the 0-th order operator of multiplication by $(\partial^2\varphi/\partial z_j \partial\bar{z}_k)$. This takes care of the sum in (2.33) coming from the first term. In the second integral in (2.33) we integrate $\partial/\partial\bar{z}_k$ by parts in the first term. Again, the boundary term vanishes (since $\sum_k \overline{(\partial\rho/\partial z_k) u_{kK}} = 0$ on the boundary, for all K). We obtain

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial a}{\partial z_j} \frac{\partial u_{jK}}{\partial \bar{z}_k} \overline{u_{kK}} e^{-\varphi}\, dV \\ = - \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 a}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} e^{-\varphi}\, dV \\ + \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial a}{\partial z_j} u_{jK} \overline{\left(- \frac{\partial u_{kK}}{\partial z_k} + \frac{\partial \varphi}{\partial z_k} u_{kK} \right)} e^{-\varphi}\, dV. \tag{2.34}$$

Because

$$\sum_{|K|=q-1}{}' \left(\sum_{k=1}^{n} \left(-\frac{\partial u_{kK}}{\partial z_k} + \frac{\partial \varphi}{\partial z_k} u_{kK} \right) \right) d\bar{z}_K \\ = \sum_{|K|=q-1}{}' \left(\sum_{k=1}^{n} -\delta_k u_{kK} \right) d\bar{z}_K \\ = \bar{\partial}^*_\varphi u \tag{2.35}$$

(compare (2.21)), the contribution from the second integral in (2.33) is

$$2\,\mathrm{Re} \left(\sum_{|K|=q-1}{}' \sum_{j=1}^{n} u_{jK} \frac{\partial a}{\partial z_j}\, d\bar{z}_K, \bar{\partial}^*_\varphi u \right)_\varphi - \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 a}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} e^{-\varphi}\, dV. \tag{2.36}$$

This completes the proof of Proposition 2.4. □

2.7 The role of pseudoconvexity

Let us recall the definition of pseudoconvexity. If Ω is a domain with C^2 boundary, Ω is called pseudoconvex if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(z) w_j \overline{w_k} \geq 0, \quad z \in b\Omega,\ w \in \mathbb{C}^n, \text{ with} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j = 0. \tag{2.37}$$

Here ρ is a (C^2) defining function for Ω. In other words, the complex Hessian of ρ should be nonnegative on vectors that are orthogonal (in the Hermitian inner product in $\mathbb{C}^n$) to the complex normal $(\partial\rho/\partial\bar{z}_1, \ldots, \partial\rho/\partial\bar{z}_n)$. The vectors w satisfying $\sum_{j=1}^{n}(\partial\rho/\partial z_j)(z)w_j = 0$ are called complex tangent vectors at $z \in b\Omega$; they make up the complex tangent space to $b\Omega$ at z, denoted by $T_z^{\mathbb{C}}(b\Omega)$. The reason for the terminology is that $w \in T_z^{\mathbb{C}}(b\Omega)$ if and only if both w and iw are tangent to $b\Omega$ at z (iw is defined as usual, i.e., $(iw)_j = iw_j, 1 \leq j \leq n$). The quadratic form appearing in (2.37), i.e., the restriction to $T_z^{\mathbb{C}}(b\Omega)$ of the complex Hessian of ρ, is called "the" Levi form of $b\Omega$ at z. Although this form depends on the choice of defining function, its essential properties do not: a different choice of ρ results in a Levi form that is a positive scalar multiple. In particular, the condition in (2.37) (and therefore our definition of pseudoconvexity) is independent of the defining function. If the inequality in (2.37) is strict, we call the domain strictly pseudoconvex. In general, without boundary regularity assumption, a domain Ω is defined to be pseudoconvex if it can be exhausted by an increasing sequence of strictly pseudoconvex domains: $\Omega = \bigcup_n \Omega_n$ with Ω_n relatively compact in Ω_{n+1} and with Ω_n a strictly pseudoconvex domain. This is compatible with our definition in the case of C^2 domains because any such domain can be exhausted by strictly pseudoconvex domains ([207], [259]). When the boundary is C^3, more can be said.

The following lemma from [191] says not only that there always exists a defining function whose level sets inside Ω are strictly pseudoconvex, but it gives an explicit form, starting from an arbitrary defining function. This lemma will be needed in Chapter 5.

Lemma 2.5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^3 defining function ρ. There is $M > 0$ such that the subdomains Ω_δ defined by $\Omega_\delta := \{z \in \Omega \mid \rho(z) + \delta e^{M|z|^2} < 0\}$ are strictly pseudoconvex subdomains of Ω. Equivalently: the defining function for Ω, $r := e^{-M|z|^2}\rho(z)$ has strictly pseudoconvex level sets $\{r = -\delta\}$ for $\delta > 0$ close enough to 0.*

Proof. The proof comes from [191]. We have

$$\frac{\partial r}{\partial z_j} = e^{-M|z|^2}\left(\frac{\partial \rho}{\partial z_j} - M\rho\overline{z_j}\right) \tag{2.38}$$

and

$$
\begin{aligned}
\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} &= -M\overline{z_j}e^{-M|z|^2}\left(\frac{\partial\rho}{\partial\bar{z}_k} - M\rho z_k\right) \\
&\quad + e^{-M|z|^2}\left(\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k} - M\frac{\partial\rho}{\partial z_j}z_k - M\rho\delta_{jk}\right).
\end{aligned} \tag{2.39}
$$

We need to check the Hessian of r on vectors w satisfying

$$
\sum_{j=1}^{n}\frac{\partial r}{\partial z_j}w_j = e^{-M|z|^2}\sum_{j=1}^{n}\left(\frac{\partial\rho}{\partial z_j} - M\rho\overline{z_j}\right)w_j = 0, \tag{2.40}
$$

i.e., w is a complex tangent to $b\Omega_\delta$. We obtain for such w

$$
\begin{aligned}
\sum_{j,k}\frac{\partial^2 r}{\partial z_j\partial\bar{z}_k}w_j\overline{w_k} &= e^{-M|z|^2}\Big\{\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}w_j\overline{w_k} \\
&\quad - M\Big(\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}w_j\Big)\Big(\sum_{k=1}^{n}z_k\overline{w_k}\Big) - M\rho|w|^2\Big\},
\end{aligned} \tag{2.41}
$$

where we have already used (2.40) once. Using (2.40) again, we obtain for the right-hand side of (2.41) (we omit the positive factor $e^{-M|z|^2}$)

$$
\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}w_j\overline{w_k} - M^2\rho\Big|\sum_{k=1}^{n}z_k\overline{w_k}\Big|^2 - M\rho|w|^2. \tag{2.42}
$$

Note that we are inside Ω, so that $-\rho > 0$. Split $w = w_T + w_N$, where w_T and w_N denote the tangential and normal parts of w respectively, with respect to the level set of ρ (not of r). Then

$$
\begin{aligned}
\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}w_j\overline{w_k} &= \sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}((w_T)_j + (w_N)_j)(\overline{(w_T)_k + (w_N)_k}) \\
&\geq \sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(w_T)_j\overline{(w_T)_k} - \Big|\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(w_N)_j\overline{(w_N)_k}\Big| \\
&\quad - 2\Big|\operatorname{Re}\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(w_T)_j\overline{(w_N)_k}\Big|.
\end{aligned} \tag{2.43}
$$

There is a constant C such that the first term on the right-hand side of (2.43) is at least $C\rho|w_T|^2$ (because Ω is pseudoconvex). Using that $|w_N| = |\sum_j \frac{\partial\rho}{\partial z_j}w_j| = M\rho|\sum_j \overline{z_j}w_j|$ (from (2.40)), we estimate (2.43) from below (changing C if necessary) by

$$
\begin{aligned}
C\rho|w|^2 - C|w_N|^2 - C|w_T||w_N| \\
&\geq C\rho|w|^2 - CM^2\rho^2\Big|\sum_j \bar{z}_j w_j\Big|^2 \\
&\quad + \frac{M}{2}\rho|w|^2 + \frac{2C^2}{M\rho}M^2\rho^2\Big|\sum_j \bar{z}_j w_j\Big|^2 \qquad (2.44)\\
&\geq \Big(C + \frac{M}{2}\Big)\rho|w|^2 - CM^2\rho^2\Big|\sum_j \bar{z}_j w_j\Big|^2 + 2C^2M\rho\Big|\sum_j \bar{z}_j w_j\Big|^2.
\end{aligned}
$$

Now we can see that if M is chosen big enough the two (positive) terms in (2.42) involving a factor $(-\rho)$ will dominate the terms in (2.44) (all of which are negative): if M is big enough $M(-\rho)|w|^2 \geq \big(C + \frac{M}{2}\big)\,|\rho||w|^2$, and

$$M^2(-\rho)\Big|\sum_{j=1}^n z_k\overline{w_k}\Big|^2 \geq CM^2\rho^2\Big|\sum_{j=1}^n z_n\overline{w_k}\Big|^2 2C^2M(-\rho)\Big|\sum_j \bar{z}_j w_j\Big|^2,$$

if also ρ is small enough. But ρ small enough is equivalent to δ small enough (once M is chosen). This completes the proof of Lemma 2.5. □

Remark. The Levi form has a very useful geometric interpretation: it computes the normal $(1,0)$-component of commutators. More precisely, if $L_1 = \sum_{j=1}^n a_j\partial/\partial z_j$ and $L_2 = \sum_{j=1}^n b_j\partial/\partial z_j$ are two complex tangential fields, then

$$\partial\rho([L_1,\overline{L_2}]) = -\sum_{j=1}^n(\overline{L_2}a_j)\frac{\partial\rho}{\partial z_j} = -\sum_{j,k=1}^n \overline{b_k}\left(\frac{\partial a_j}{\partial \bar{z}_k}\right)\frac{\partial\rho}{\partial z_j} = \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}a_j\overline{b_k}. \tag{2.45}$$

In the last equality we have used the fact that L_1 is complex tangential, that is, that $\sum_{j=1}^n a_j(\partial\rho/\partial z_j) = 0$. For more information on the Levi form, we refer the reader to [54], Section 10.3, [92], Section 3.1.1.

Pseudoconvexity is a central notion in several complex variables, and there are many equivalent characterizations. We refer the reader to [172], [207], [259], [81]. In our context, the motivation for the definition comes from (2.24): for $u \in \mathrm{dom}(\bar\partial^*)$, $\sum_j(\partial\rho/\partial z_j)u_{jK} = 0$ on $b\Omega$ for all K, so that pseudoconvexity implies that the boundary term in (2.24) is nonnegative. This turns out to be essential.

We keep the differentiability assumptions from Proposition 2.4 on Ω and on $u \in \mathrm{dom}(\bar\partial^*)$. Choosing $\varphi \equiv 0$ and $a \equiv 1$ in (2.24) gives (because the boundary integral is nonnegative)

$$\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \geq \sum_J{}' \sum_{j=1}^n \left\|\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2; \tag{2.46}$$

that is, the bar derivatives of u are controlled (in $\mathcal{L}^2(\Omega)$) by $\bar\partial u$ and $\bar\partial^* u$. Since integration by parts turns tangential vector fields of type $(1,0)$ into fields of type $(0,1)$, one obtains that for many purposes, derivatives of type $(1,0)$ that are (complex) tangential are also 'under control'; see Lemma 5.6 and the remark preceding Theorem 3.6 for precise statements and applications.

Now let $a = 1 - e^b$, where $b \in C^2(\overline{\Omega})$ is nonpositive (so that $1 - e^b \geq 0$). (2.24) gives (again because the boundary integral is nonnegative)

$$\begin{aligned}\|\sqrt{a}\,\bar\partial u\|^2 + \|\sqrt{a}\,\bar\partial^* u\|^2 &\geq -2\,\mathrm{Re}\Big(e^b {\sum_K}' \Big(\sum_{j=1}^n u_{jK}\frac{\partial b}{\partial z_j}\Big)d\bar z_K, \bar\partial^* u\Big) \\ &\quad + {\sum_K}' \sum_{j,k=1}^n \int_\Omega e^b \left(\frac{\partial b}{\partial z_j}\frac{\partial b}{\partial \bar z_k} + \frac{\partial^2 b}{\partial z_j \partial \bar z_k}\right) u_{jK}\overline{u_{kK}}\, dV \\ &\geq -\|e^{b/2}\bar\partial^* u\|^2 + {\sum_K}' \sum_{j,k=1}^n \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar z_k} u_{jK}\overline{u_{kk}}\, dV.\end{aligned} \tag{2.47}$$

The last inequality follows by applying the Cauchy–Schwarz inequality and then the inequality $2|cd| \leq c^2 + d^2$ to the term with Re. Taking $\|e^{b/2}\bar\partial^* u\|^2$ to the other side and noting that $a + e^b = 1$ proves the following lemma when $u \in C^1_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar\partial^*)$:

Lemma 2.6. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary, let $b \in C^2(\overline{\Omega})$, $b \leq 0$. Then*

$$\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \geq {\sum_{|K|=q-1}}' \sum_{j,k=1}^n \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar z_k} u_{jK}\overline{u_{kK}}\, dV, \tag{2.48}$$

for all $u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$.

Proof. When u is only assumed in $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$, we use Proposition 2.3 to reduce to the case where $u \in C^1_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar\partial^*)$, shown above. □

Remark. (2.46) similarly holds for all $u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$: if $u_n \to u$ in the graph norm, then $\partial(u_n)_J/\partial \bar z_j \to \partial u_J/\partial \bar z_j$ as distributions. Since the $\mathcal{L}^2$-norms remain bounded, $\partial u_J/\partial \bar z_j \in \mathcal{L}^2(\Omega)$, for all J, j, and (2.46) holds. Moreover, both (2.46) and (2.48) hold on any bounded pseudoconvex domain, without any regularity assumption on the boundary. This will be established in Corollary 2.13 by exploiting the $\bar\partial$-Neumann operators on exhausting smooth subdomains.

We now choose a particular function $b(z)$ in (2.48). Fix $P \in \Omega$ and set $b(z) := -1 + |z - P|^2/D^2$, where D is the diameter of Ω. Then we have $e^{b(z)} \geq e^{-1}$ and $(\partial^2 b/\partial z_j \partial \bar z_k)(z) = \delta_{jk}/D^2$. (2.48) gives in this case

$$\|u\|^2 \leq \frac{D^2 e}{q}(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2), \quad u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*). \tag{2.49}$$

The factor $1/q$ arises because each strictly increasing multi-index J of length q will appear q times in the sum on the right-hand side of (2.48).

We can dispense with the smoothness requirement on $b\Omega$ (needed in our argument for integration by parts). If Ω is only assumed bounded and pseudoconvex, it can be exhausted by an increasing union of pseudoconvex domains of class C^2. We thus need to see that (2.49) is preserved under such unions. The reason that this is not obvious is that when a form u in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ on Ω is restricted to a subdomain, it need not be in $\mathrm{dom}(\bar{\partial}^*)$ there (as can be seen from the boundary condition (2.7) for membership in $\mathrm{dom}(\bar{\partial}^*)$). It is convenient to first reformulate (2.49). For this, we make no assumptions about boundary regularity of Ω. As the reader will see, it is advisable here to use subscripts to indicate the levels of the forms on which the operators act.

Assume (2.49). Note that $\ker(\bar{\partial}_q)^{\perp} = \overline{\mathrm{Im}(\bar{\partial}_q^*)} \subseteq \ker(\bar{\partial}_{q-1}^*)$. But if $u \in \ker(\bar{\partial}_{q-1}^*) \cap (\ker(\bar{\partial}_q)^{\perp})^{\perp} = \ker(\bar{\partial}_{q-1}^*) \cap \ker(\bar{\partial}_q)$, then by (2.49) $u = 0$. So $\ker(\bar{\partial}_q)^{\perp} = \ker(\bar{\partial}_{q-1}^*)$, i.e., $\mathcal{L}^2_{(0,q)}(\Omega) = \ker(\bar{\partial}_q) \oplus \ker(\bar{\partial}_{q-1}^*)$. (2.49) now implies that $\mathrm{Im}(\bar{\partial}_q)$ and $\mathrm{Im}(\bar{\partial}_{q-1}^*)$ are closed. Hence $\mathrm{Im}(\bar{\partial}_q^*)$ and $\mathrm{Im}(\bar{\partial}_{q-1})$ are closed as well (note that $(\bar{\partial}_{q-1}^*)^* = \bar{\partial}_{q-1}$): this is a general fact from functional analysis; it follows from an estimation like (2.51) below. Therefore, $\ker(\bar{\partial}_q) = \mathrm{Im}(\bar{\partial}_{q-1})$ and $\ker(\bar{\partial}_{q-1}^*) = \mathrm{Im}(\bar{\partial}_q^*)$. Consequently, by the above orthogonal decomposition of $\mathcal{L}^2_{(0,q)}(\Omega)$, every $u \in \mathcal{L}^2_{(0,q)}(\Omega)$ can be written as

$$u = \bar{\partial}_{q-1} v + \bar{\partial}_q^* w, \quad v \in \ker(\bar{\partial}_{q-1})^{\perp},\ w \in \ker(\bar{\partial}_q^*)^{\perp}. \tag{2.50}$$

To estimate the norm of v, it suffices to pair with forms in $\mathrm{Im}(\bar{\partial}_{q-1}^*)$ (since these are dense in $\ker(\bar{\partial}_{q-1})^{\perp}$). Let $\alpha \in \mathrm{dom}(\bar{\partial}_{q-1}^*) \cap \ker(\bar{\partial}_{q-1}^*)^{\perp} \subseteq \ker(\bar{\partial}_q)$. Then

$$\left|(v, \bar{\partial}_{q-1}^*\alpha)\right|^2 = \left|(\bar{\partial}_{q-1} v, \alpha)\right|^2 \le \|\bar{\partial}_{q-1} v\|^2 \left(\frac{D^2 e}{q}\right) \|\bar{\partial}_{q-1}^*\alpha\|^2. \tag{2.51}$$

In the last inequality in (2.51), we have used (2.49) for α, which is in $\ker(\bar{\partial}_q)$. Combining the resulting estimate for $\|v\|^2_{\mathcal{L}^2_{(0,q-1)}(\Omega)}$ with the analogous estimate for $\|w\|^2_{\mathcal{L}^2_{(0,q+1)}(\Omega)}$ gives

$$\|v\|^2_{\mathcal{L}^2_{(0,q-1)}(\Omega)} + \|w\|^2_{\mathcal{L}^2_{(0,q+1)}(\Omega)} \le \frac{D^2 e}{q} \|u\|^2 \tag{2.52}$$

(since the decomposition (2.50) is orthogonal). Thus (2.49) implies that every $u \in \mathcal{L}^2_{(0,q)}(\Omega)$ can be written as in (2.50), with estimate (2.52).

Conversely, assume that every $u \in \mathcal{L}^2_{(0,q)}(\Omega)$ can be written as in (2.50), with estimate (2.52). Let $u \in \mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}_{q-1}^*)$, and write $u = \bar{\partial}_{q-1} v + \bar{\partial}_q^* w$, with (2.52). Note that then $\bar{\partial}_{q-1} v \in \mathrm{dom}(\bar{\partial}_{q-1}^*)$ and $\bar{\partial}_q^* w \in \mathrm{dom}(\bar{\partial}_q)$ (since $\bar{\partial}_q^* w \in$

$\ker(\bar\partial^*_{q-1})$ and $\bar\partial_{q-1}v \in \ker(\bar\partial_q)$). Therefore,

$$\begin{aligned}\|u\|^2 &= \|\bar\partial_{q-1}v\|^2 + \|\bar\partial^*_q w\|^2 = (\bar\partial^*_{q-1}\bar\partial_{q-1}v, v) + (\bar\partial_q\bar\partial^*_q w, w)\\ &\le \|\bar\partial^*_{q-1}\bar\partial_{q-1}v\|\|v\| + \|\bar\partial_q\bar\partial^*_q w\|\|w\|\\ &\le \left(\|\bar\partial^*_{q-1}\bar\partial_{q-1}v\|^2 + \|\bar\partial_q\bar\partial^*_q w\|^2\right)^{1/2}\left(\|v\|^2+\|w\|^2\right)^{1/2}\\ &\le \left(\|\bar\partial^*_{q-1}u\|^2 + \|\bar\partial_q u\|^2\right)^{1/2}\left(\frac{D^2e}{q}\right)^{1/2}\|u\|.\end{aligned} \tag{2.53}$$

In the last inequality in (2.53) we have used that $\bar\partial^*_{q-1}\bar\partial_{q-1}v = \bar\partial^*_{q-1}u$, that $\bar\partial_q\bar\partial^*_q w = \bar\partial_q u$, and (2.52). (2.53) immediately gives (2.49).

We have shown: (2.49) holds for all $u \in \mathrm{dom}(\bar\partial_q)\cap\mathrm{dom}(\bar\partial^*_{q-1})$ if and only if every $u \in \mathcal{L}^2_{(0,q)}(\Omega)$ has a representation (2.50) satisfying estimate (2.52). We now use this reformulation to prove that (2.49) holds on any bounded pseudoconvex domain.

Proposition 2.7. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, with diameter D. Then for $u \in \mathrm{dom}(\bar\partial)\cap\mathrm{dom}(\bar\partial^*) \subseteq \mathcal{L}^2_{(0,q)}(\Omega)$, we have the estimate*

$$\|u\|^2 \le \frac{D^2e}{q}(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2), \quad 1 \le q \le n. \tag{2.54}$$

Note that all the ramifications of (2.49) discussed above consequently also hold on any bounded pseudoconvex domain.

Proof. It suffices to see that (2.50), (2.52) are preserved under increasing unions. So let $\Omega = \bigcup_n \Omega_n$, where $\overline{\Omega}_n \subset\subset \Omega_{n+1}$, and each Ω_n is a pseudoconvex domain with C^2 boundary. Let $u \in \mathcal{L}^2_{(0,q)}(\Omega)$. Then $u|_{\Omega_n} \in \mathcal{L}^2_{(0,q)}(\Omega_n)$ and

$$u|_{\Omega_n} = \bar\partial v_n + \bar\partial^* w_n \tag{2.55}$$

with

$$\|v_n\|^2_{\mathcal{L}^2_{(0,q-1)}(\Omega_n)} + \|w_n\|^2_{\mathcal{L}^2_{(0,q+1)}(\Omega_n)} \le \frac{D^2e}{q}\|u\|^2. \tag{2.56}$$

We have used here that Ω_n has diameter $\le D$ and that $\|u|_{\Omega_n}\|^2_{\mathcal{L}^2_{(0,q)}(\Omega_n)} \le \|u\|_{\mathcal{L}^2_{(0,q)}(\Omega)}$. Extending the v_n's and the w_n's by zero outside of Ω_n gives two sequences bounded in $\mathcal{L}^2_{(0,q-1)}(\Omega)$ and $\mathcal{L}^2_{(0,q+1)}(\Omega)$, respectively. Passing to an appropriate subsequence gives two weak limits v and w with $\|v\|^2_{\mathcal{L}^2_{(0,q-1)}(\Omega)} + \|w\|^2_{\mathcal{L}^2_{(0,q+1)}(\Omega)} \le \frac{D^2e}{q}\|u\|^2$. Because the decomposition in (2.55) is orthogonal, $\bar\partial v_n|_{\Omega_n}$ is bounded in $\mathcal{L}^2$ independently of u. This and the fact that weak limit and distributional limits agree shows that $v \in \mathrm{dom}(\bar\partial)$ and a subsequence of $\{\bar\partial v_n\}_{n=1}^\infty$ (extended by zero) converge to $\bar\partial v$ weakly. To see that $w \in \mathrm{dom}(\bar\partial^*_q)$, observe that for $\alpha \in \mathrm{dom}(\bar\partial_q)$,

$$\begin{aligned}|(w, \bar\partial\alpha)| &\le \limsup_{n\to\infty}|(w_n, \bar\partial\alpha)| = \limsup_{n\to\infty}|(w_n, \bar\partial\alpha)_{\Omega_n}|\\ &= \limsup_{n\to\infty}|(\bar\partial^* w_n, \alpha)_{\Omega_n}| \le (\limsup_{n\to\infty}\|\bar\partial^* w_n\|)\|\alpha\| \le \|u\|\|\alpha\|.\end{aligned} \tag{2.57}$$

We have again used that the decomposition (2.55) is orthogonal, so that $\|\bar{\partial}^* w_n\| \le \|u|_{\Omega_n}\| \le \|u\|$. Weak convergence of a subsequence of $\{\bar{\partial}^* w_n\}_{n=1}^{\infty}$ to $\bar{\partial}^* w$ follows now also along the lines of (2.57). Consequently, $u = \bar{\partial}v + \bar{\partial}^* w$, and v and w satisfy the required estimate. This completes the proof of Proposition 2.7. □

It is worth noting that the constant in (2.54) does not depend on the dimension n. Also note that as a result of (2.54), we may take the quadratic form on the right-hand side of (2.54) as the square of the graph norm for $\bar{\partial} \oplus \bar{\partial}^*$ in a bounded pseudoconvex domain.

2.8 Complex Laplacian and $\bar{\partial}$-Neumann operator

The quadratic form in the previous section, and the role it plays, suggest looking at the self-adjoint (unbounded) operator defined by it. It will be convenient to have the notation

$$Q(u, v) := (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^* u, \bar{\partial}^* v) \tag{2.58}$$

for $u, v \in \text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$, $1 \le q \le n$. With the inner product (2.58), $\text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$ is a Hilbert space (in view of Proposition 2.7 and the fact that $\bar{\partial}_q$ and $\bar{\partial}^*_{q-1}$ are closed operators). Therefore, there is a unique selfadjoint operator on $\mathcal{L}^2_{(0,q)}(\Omega)$ associated to the quadratic form Q, compare for example Theorem VIII.15 in [260] or Theorem 4.4.2 in [96]. This operator is the complex Laplacian $\Box_q$ associated to the $\bar{\partial}$-complex. We recall the construction from [260]. Denote by j the natural embedding of $\text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$ into $\mathcal{L}^2_{(0,q)}(\Omega)$, and by $\hat{j}$ the conjugate of its adjoint from $\mathcal{L}^2_{(0,q-1)}(\Omega)$ into $\bar{X}$, the space of conjugate linear functionals on $\text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$. That is, if $g \in \mathcal{L}^2_{(0,q)}(\Omega)$, then $\langle \hat{j}(g), u\rangle := (g, u)_{\mathcal{L}^2_{(0,q)}(\Omega)}$, where $\langle \cdot, \cdot \rangle$ is the pairing between $\bar{X}$ and $\text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$. Next, consider the usual isometric identification of a Hilbert space with its conjugate dual: define an operator $\tilde{\Box}_q$ from $\text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$ onto $\bar{X}$ as follows. For $u, v \in \text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$, set

$$\langle \tilde{\Box}_q v, u\rangle := (\bar{\partial}v, \bar{\partial}u) + (\bar{\partial}^* v, \bar{\partial}^* u). \tag{2.59}$$

Then, by elementary Hilbert space theory, $\tilde{\Box}_q$ is an isometric isomorphism from $\text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1})$ onto $\bar{X}$. Now set

$$\text{dom}(\Box_q) := \{v \in \text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1}) / \tilde{\Box}_q v \in \mathcal{L}^2_{(0,q)}(\Omega)\}, \tag{2.60}$$

and, for $v \in \text{dom}(\Box_q)$,

$$\Box_q v := \tilde{\Box}_q v. \tag{2.61}$$

Note that in (2.60), we are slightly abusing the notation: $\tilde{\Box}_q v$ is an element of $\bar{X}$, not of $\mathcal{L}^2_{(0,q-1)}(\Omega)$; what is meant is that $\tilde{\Box}_q v \in \hat{j}(\mathcal{L}^2_{(0,q)}(\Omega)) \subseteq \bar{X}$. Since $\hat{j}$ is injective (because $\mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$ is dense in $\mathcal{L}^2_{(0,q)}(\Omega)$), we can identify $\hat{j}(g)$ with $g \in \mathcal{L}^2_{(0,q)}(\Omega)$. It follows that $v \in \mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$ belongs to $\mathrm{dom}(\Box_q)$ if and only if there exists a (necessarily unique) form $\Box_q v \in \mathcal{L}^2_{(0,q)}(\Omega)$ such that if $u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$, then

$$(\bar{\partial}v, \bar{\partial}u) + (\bar{\partial}^* v, \bar{\partial}^* u) = (\Box_q v, u)_{\mathcal{L}^2_{(0,q)}(\Omega)}. \tag{2.62}$$

Alternatively, $\Box_q$ can be described as follows.

Proposition 2.8. *Let $1 \leq q \leq n$. Then*

$$\mathrm{dom}(\Box_q) = \{u \in \mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1}) / \bar{\partial}_q u \in \mathrm{dom}(\bar{\partial}^*_q), \bar{\partial}^*_{q-1} u \in \mathrm{dom}(\bar{\partial}_{q-1})\}, \tag{2.63}$$

and for $u \in \mathrm{dom}(\Box_q)$,

$$\Box_q u = \bar{\partial}\bar{\partial}^* u + \bar{\partial}^* \bar{\partial} u. \tag{2.64}$$

Proof. It is clear that if u is as in the right-hand side of (2.63), then $u \in \mathrm{dom}(\Box_q)$, and $\Box_q u = \bar{\partial}\bar{\partial}^* u + \bar{\partial}^* \bar{\partial} u$. It remains to be shown that $\mathrm{dom}(\Box_q)$ is contained in the right-hand side of (2.63), that is, that $\bar{\partial}_q u \in \mathrm{dom}(\bar{\partial}^*_q)$ and $\bar{\partial}^*_{q-1} u \in \mathrm{dom}(\bar{\partial}_{q-1})$ when $u \in \mathrm{dom}(\Box_q)$. So let $u \in \mathrm{dom}(\Box_q)$, $v \in \mathrm{dom}(\bar{\partial}_q)$. Then, because $(I - P_q)v \in \ker(\bar{\partial}^*_{q-1})$,

$$\begin{aligned}(\bar{\partial}_q u, \bar{\partial}_q v) &= (\bar{\partial}_q u, \bar{\partial}_q (I - P_q)v) \\ &= (\bar{\partial}_q u, \bar{\partial}_q (I - P_q)v) + (\bar{\partial}^*_{q-1} u, \bar{\partial}^*_{q-1}(I - P_q)v) \\ &= (\Box_q u, (I - P_q)v) = ((I - P)\Box_q u, v).\end{aligned} \tag{2.65}$$

The penultimate equality in (2.65) follows from (2.62), because $u \in \mathrm{dom}(\Box_q)$. (2.65) shows that $\bar{\partial} u \in \mathrm{dom}(\bar{\partial}^*_q)$, and

$$\bar{\partial}^*_q \bar{\partial}_q u = (I - P_q)\Box_q u. \tag{2.66}$$

On the other hand, if $v \in \mathrm{dom}(\bar{\partial}^*_{q-1})$, then $\bar{\partial}^*_{q-1} v = \bar{\partial}^*_{q-1} P_q v$, and so

$$\begin{aligned}(\bar{\partial}^*_{q-1} u, \bar{\partial}^*_{q-1} v) &= (\bar{\partial}^*_{q-1} u, \bar{\partial}^*_{q-1} P_q v) + (\bar{\partial}_q u, \bar{\partial}_q P_q v) \\ &= (\Box_q u, P_q v) = (P_q \Box_q u, v).\end{aligned} \tag{2.67}$$

(2.67) shows that $\bar{\partial}^*_{q-1} u \in \mathrm{dom}(\bar{\partial}^*_{q-1})^* = \mathrm{dom}(\bar{\partial}_{q-1})$ (since $(\bar{\partial}^*_{q-1})^* = \bar{\partial}_{q-1}$), and that

$$\bar{\partial}_{q-1} \bar{\partial}^*_{q-1} u = P_q \Box_q u. \tag{2.68}$$

This completes the proof of Proposition 2.8. $\Box$

The two boundary conditions $u \in \mathrm{dom}(\bar\partial^*)$ and $\bar\partial u \in \mathrm{dom}(\bar\partial^*)$ are called the $\bar\partial$-Neumann boundary conditions. The second one, $\bar\partial u \in \mathrm{dom}(\bar\partial^*)$, is the so-called 'free boundary condition'. It is only implicit in the definition of $\Box_q$ via the quadratic form Q, while the condition $u \in \mathrm{dom}(\bar\partial^*)$ is explicit (it enters into the definition of $\mathrm{dom}(\Box_q)$). Note that by (2.8), (2.15), $u \in \mathrm{dom}(\bar\partial^*)$ amounts to a Dirichlet boundary condition on the normal component of u, and $\bar\partial u \in \mathrm{dom}(\bar\partial^*)$ amounts to a Dirichlet boundary condition on the normal component of $\bar\partial u$. i.e., a complex (or $\bar\partial$) Neumann condition.

The $\bar\partial$-Neumann problem is the study of 'the' operator $\Box_q$, $1 \le q \le n$. The fundamental $\mathcal{L}^2$-existence theorem for the $\bar\partial$-Neumann problem says that $\Box_q$ is invertible on a bounded pseudoconvex domain. This inverse is the $\bar\partial$-Neumann operator N_q. Recall that $j = j_q$ denotes the embedding $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega)$, where $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$ is provided with the graph norm, and that by Proposition 2.7, this embedding is continuous.

Theorem 2.9. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ where $n \ge 2$. Let D denote the diameter of Ω, and suppose $1 \le q \le n$.*

(1) *The complex Laplacian $\Box_q = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is an unbounded selfadjoint surjective operator on $\mathcal{L}^2_{(0,q)}(\Omega)$ having a bounded selfadjoint inverse $N_q = j_q \circ j_q^*$.*

(2) *For all $u \in \mathcal{L}^2_{(0,q)}(\Omega)$, we have the estimates*

$$\begin{aligned} \|N_q u\| &\le \frac{D^2 e}{q}\|u\|, \\ \|\bar\partial N_q u\|^2 + \|\bar\partial^* N_q u\|^2 &\le \frac{D^2 e}{q}\|u\|^2. \end{aligned} \tag{2.69}$$

(3)

$$\begin{aligned} \bar\partial N_q u &= N_{q+1}\bar\partial u, u \in \mathrm{dom}(\bar\partial) \\ \bar\partial^* N_q u &= N_{q-1}\bar\partial^* u, u \in \mathrm{dom}(\bar\partial^*), \quad q \ge 2. \end{aligned} \tag{2.70}$$

(4)

$$P_{q-1}u = u - \bar\partial^* N_q \bar\partial u. \tag{2.71}$$

Remarks. (i) The restriction in the second equation in (2.70) that $q \ge 2$ is due to the fact that we have not defined a $\bar\partial$-Neumann operator when $q = 0$. However, this can be done; one only has to take into account that $\Box_0 = \bar\partial_0^*\bar\partial_0$ now only maps onto $\mathrm{Im}(\bar\partial_0^*) = \ker(\bar\partial_0)^\perp$. Details may be found in [81] Theorem 4.4.3.

(ii) Note that in (2.70), the operators $N_{q+1}\bar\partial_q$ and $N_{q-1}\bar\partial_{q-1}^*$, originally defined only on $\mathrm{dom}(\bar\partial_q)$ and $\mathrm{dom}(\bar\partial_{q-1}^*)$, respectively, extend to bounded operators on all of $\mathcal{L}^2_{(0,q)}(\Omega)$.

Corollary 2.10. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, where $n \geq 2$, and assume $1 \leq q \leq n$.*

(1) *If $u \in \ker(\bar\partial_q)$, then $\bar\partial^*_{q-1} N_q u$ gives the solution f to the equation $\bar\partial_{q-1} f = u$ of minimal $\mathcal{L}^2_{(0,q-1)}(\Omega)$-norm.*

(2) *If $u \in \ker(\bar\partial^*_{q-1})$ then $\bar\partial_q N_q u$ gives the solution f to the equation $\bar\partial^*_q f = u$ of minimal $\mathcal{L}^2_{(0,q+1)}(\Omega)$-norm.*

The solutions of minimal $\mathcal{L}^2$-norm to the $\bar\partial$ and $\bar\partial^*$ equations, respectively, provided in Corollary 2.10, are often referred to as the canonical, or Kohn, solutions. In turn, one can express N_q in terms of these solution operators (see [258], compare also [125], p. 55). Let $u \in \mathcal{L}^2_{(0,q)}(\Omega)$, $1 \leq q \leq n$. Then

$$\begin{aligned} N_q u &= N_q(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)N_q u \\ &= (N_q\bar\partial)(\bar\partial^* N_q)u + (N_q\bar\partial^*)(\bar\partial N_q)u \\ &= (\bar\partial^* N_q)^*(\bar\partial^* N_q)u + (\bar\partial^* N_{q+1})(\bar\partial^* N_{q+1})^* u. \end{aligned} \tag{2.72}$$

In the last equality, we have used (2.70).

Proof of Corollary 2.10. We start with $u = \bar\partial\bar\partial^* N_q u + \bar\partial^*\bar\partial N_q u$. If $\bar\partial u = 0$, then $\bar\partial^*\bar\partial N_q u \in \ker(\bar\partial_q)$ as well. But $\bar\partial^*\bar\partial N_q u$ is also orthogonal to $\ker(\bar\partial_q)$. Therefore, $\bar\partial^*\bar\partial N_q u = 0$, and $u = \bar\partial(\bar\partial^* N_q u)$. A similar argument proves part (2) of Corollary 2.10. It is clear that these solutions have minimal norm in $\mathcal{L}^2_{(0,q)}(\Omega)$, as they are orthogonal to the respective kernels. □

Proof of Theorem 2.9. If N_q is to invert $\Box_q$, it must map into $\mathrm{dom}(\Box_q) \subseteq \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$, and we must have

$$\begin{aligned} (u, v) = (\Box_q N_q u, v) &= \big((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)N_q u, v\big) \\ &= (\bar\partial^* N_q u, \bar\partial^* v) + (\bar\partial N_q u, \bar\partial v), \quad v \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*). \end{aligned} \tag{2.73}$$

This means that as an operator to $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$, N_q must coincide with j_q^* (note that $(u, v) = (u, j_q v)$). As an operator to $\mathcal{L}^2_{(0,q)}(\Omega)$, N_q must thus equal $j_q \circ j_q^*$. Accordingly, we set $N_q = j_q \circ j_q^*$. It remains to check that N_q so defined has all the required properties.

We have $N_q^* = (j_q \circ j_q^*)^* = j_q \circ j_q^* = N_q$, i.e., N_q is selfadjoint (it is of course bounded). Also, estimate (2.54) in Proposition 2.7 gives

$$\begin{aligned} \|\bar\partial N_q u\|^2 + \|\bar\partial^* N_q u\|^2 &= \|j_q^* u\|^2_{\text{graph}} \\ &\leq \|j_q^*\|^2 \|u\|^2 = \|j_q\|^2 \|u\|^2 \leq \frac{D^2 e}{q} \|u\|^2, \end{aligned} \tag{2.74}$$

and

$$\|N_q u\|^2 = \|j_q \circ j_q^* u\|^2 \le \|j_q\|^4 \|u\|^2 \le \left(\frac{D^2 e}{q}\right)^2 \|u\|^2. \tag{2.75}$$

(2.74) and (2.75) give the estimates in part (2) of Theorem 2.9.

Next, we show that N_q as defined above does indeed invert $\Box_q$. Consider $\widetilde{\Box}_q N_q u$. We have

$$\begin{aligned} \langle \widetilde{\Box}_q N_q u, v\rangle &= Q(N_q u, v) = Q(j_q^* u, v) \\ &= (u, j_q v) = (u, v), \quad v \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*). \end{aligned} \tag{2.76}$$

(2.76) says firstly that $N_q u \in \mathrm{dom}(\Box_q)$ (see (2.60)) and secondly that

$$\Box_q N_q u = u \tag{2.77}$$

(see (2.62)). Since $\ker(\Box_q) = \{0\}$ (even $\ker\widetilde{\Box}_q = \{0\}$: $\widetilde{\Box}_q$ is an isomorphism of $\mathrm{dom}(\bar\partial_q) \cap \mathrm{dom}(\bar\partial_{q-1}^*)$ onto its conjugate dual), we also get

$$N_q \Box_q u = u, \quad u \in \mathrm{dom}(\Box_q). \tag{2.78}$$

We now check the commutation properties in part (3) of Theorem 2.9. Let $u \in \mathrm{dom}(\bar\partial)$. Then

$$\begin{aligned} N_{q+1}\bar\partial u &= N_{q+1}\bar\partial(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)N_q u \\ &= N_{q+1}\bar\partial\bar\partial^*\bar\partial N_q u = N_{q+1}(\bar\partial\bar\partial^*)\bar\partial N_q \\ &= N_{q+1}(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\bar\partial N_q u = \bar\partial N_q u. \end{aligned} \tag{2.79}$$

We have used here that $\bar\partial^*\bar\partial N_q u \in \mathrm{dom}(\bar\partial_q)$ and $\bar\partial N_q u \in \mathrm{dom}(\Box_{q+1})$ if $u \in \mathrm{dom}(\bar\partial)$: $\bar\partial N_q u \in \mathrm{dom}(\bar\partial_{q+1}) \cap \mathrm{dom}(\bar\partial_q^*)$, $\bar\partial(\bar\partial N_q u) = 0$, and $\bar\partial^*\bar\partial N_q u = u - \bar\partial\bar\partial^* N_q u \in \mathrm{dom}(\bar\partial)$. The second equality in (2.70) now follows by a similar argument or by taking adjoints.

Finally, we check (2.71). Note that we are not assuming that $u \in \mathrm{dom}(\bar\partial_q)$. Rather, (2.71) should be interpreted in the sense that $\bar\partial^* N_q \bar\partial u$ extends as a continuous operator form $\mathrm{dom}(\bar\partial_q)$ (which is dense in $\mathcal{L}^2_{(0,q-1)}(\Omega)$) to $\mathcal{L}^2_{(0,q-1)}(\Omega)$. Namely, if $u \in \mathrm{dom}(\bar\partial_{q-1})$, then $\bar\partial^* N_q \bar\partial u = \bar\partial^*\bar\partial N_{q-1} u$. Since $u = \bar\partial^*\bar\partial N_{q-1} u + \bar\partial\bar\partial^* N_{q-1} u$ is an orthogonal decomposition, we have $\|\bar\partial^* N_q \bar\partial u\| = \|\bar\partial^*\bar\partial N_{q-1} u\| \le \|u\|$. To see that (2.71) holds, it now suffices to observe that $\bar\partial^* N_q \bar\partial u$ is orthogonal to $\ker(\bar\partial_{q-1})$ and that $\bar\partial(u - \bar\partial^* N_q \bar\partial u) = \bar\partial u - \bar\partial u = 0$ when $u \in \mathrm{dom}(\bar\partial_{q-1})$. Since both $\ker(\bar\partial_{q-1})$ and $\ker(\bar\partial_{q-1})^\perp$ are closed in $\mathcal{L}^2_{(0,q-1)}(\Omega)$ (by Lemma 2.1 for $\ker(\bar\partial_{q-1})$), density and continuity show that this statement holds for all $u \in \mathcal{L}^2_{(0,q-1)}(\Omega)$. This completes the proof of Theorem 2.9. □

Remarks. (i) It should be noted that the crucial result for the existence of the $\bar\partial$-Neumann operator (as a bounded operator on $\mathcal{L}^2_{(0,q)}(\Omega)$), and for its properties, as given in Theorem 2.9, is Proposition 2.7 (that is, boundedness of j_q). Indeed, once Proposition 2.7 was established, we have only used elementary Hilbert space arguments.

(ii) In establishing Theorem 2.9, we used pseudoconvexity only to ensure that the boundary term $\sum_K' \sum_{j,k=1}^n \int_{b\Omega} a(\partial^2\rho/\partial z_j \partial\bar{z}_k) u_{jK}\overline{u_{kK}} e^{-\varphi} d\sigma$ in (2.24) in Proposition 2.4 is nonnegative. In fact, pseudoconvexity insures that for *each K fixed*, the term is nonnegative, which is more than what is needed when $q > 1$. When $q > 1$, we only need that $\sum_K' \sum_{j,k=1}^n (\partial^2\rho/\partial z_j \partial\bar{z}_k) u_{jK}\overline{u_{kK}} > 0$ on $b\Omega$. For this, it suffices that the Levi form have the property that the sum of any q eigenvalues (equivalently, the sum of the smallest q eigenvalues) be nonnegative, see Lemma 4.7 in Chapter 4. Accordingly, Theorem 2.9 holds for q fixed under this weaker (if $q > 1$) hypothesis.

2.9 Applications of real potential theory

In this section, we gather some observations about the $\bar\partial$-Neumann problem that follow from the elliptic theory of the Dirichlet problem for the real Laplacian.

We first take a closer look at the operator $\bar\partial\vartheta + \vartheta\bar\partial$. On domains in $\mathbb{C}^n$, it takes a simple form: it acts coefficientwise as $-\frac{1}{4}\Delta$. In particular, then, so does $\Box_q$ on its domain. We remark that this is not true on general complex manifolds, but it continues to hold on Kähler manifolds (see for example [5] for information on Kähler manifolds).

Lemma 2.11.

$$(\bar\partial\vartheta + \vartheta\bar\partial)\sum_J{}' u_J\, d\bar{z}_J = -\frac{1}{4}\sum_J{}' (\Delta u_J)\, d\bar{z}_J. \tag{2.80}$$

Proof. The proof of the lemma results from a direct computation:

$$\begin{aligned}\bar\partial\vartheta u &= -\bar\partial\Bigg(\sum_{|K|=q-1}{}' \Bigg(\sum_{j=1}^n \frac{\partial u_{jK}}{\partial z_j}\Bigg) d\bar{z}_K\Bigg)\\ &= -\sum_{|K|=q-1}{}' \sum_{j,k=1}^n \frac{\partial^2 u_{jK}}{\partial z_j \partial\bar{z}_k}\, d\bar{z}_k \wedge d\bar{z}_K;\end{aligned} \tag{2.81}$$

$$\begin{aligned}\vartheta\bar\partial u &= \sum_J{}' \sum_{k=1}^n \vartheta\Bigg(\frac{\partial u_J}{\partial\bar{z}_k} d\bar{z}_k \wedge d\bar{z}_J\Bigg)\\ &= -\sum_{J,M}{}' \sum_{k,j=1}^n \frac{\partial}{\partial z_j}\Bigg(\Bigg(\frac{\partial u_J}{\partial\bar{z}_k} d\bar{z}_k \wedge d\bar{z}_J\Bigg)_{jM}\Bigg) d\bar{z}_M\\ &= -\sum_J{}' \sum_{j=1}^n \frac{\partial^2 u_J}{\partial z_j \partial\bar{z}_j}\, d\bar{z}_J + \sum_{|K|=q-1}{}' \sum_{k,j} \frac{\partial^2 u_{jK}}{\partial z_j \partial\bar{z}_k}\, d\bar{z}_k \wedge d\bar{z}_K.\end{aligned} \tag{2.82}$$

The first term on the right-hand side of (2.82) arises from collecting terms where $k = j$, $k \notin J$, $j \notin M$, and the second term arises from collecting terms where $k \in M$, $j \in J$. (Note that then $J\backslash\{j\} = M\backslash\{k\} = K$.) Adding (2.81) and (2.82) gives the desired result. □

Lemma 2.11 has several useful consequences that follow from the classical elliptic theory of the Laplacian. A first immediate consequence of Lemma 2.11 is the *interior elliptic regularity* of $\Box$: if $V \subset\subset U \subseteq \Omega$ are open sets, with V relatively compact in U, and $u \in W^s_{(0,q)}(U)$, then $N_q u \in W^{s+2}_{(0,q)}(V)$, $s \geq -1$. This holds because of (2.80) and because the corresponding fact holds for Δ. More precisely, the elliptic theory for the Laplacian (see e.g. [122], [215], [299], [294]) combines with (2.80) to give the following estimate: let $s \geq -1$, $t \in \mathbb{R}$; then there is a constant C such that

$$\|u\|_{W^{s+2}_{(0,q)}(V)} \leq C\left(\|\Box_q u\|_{W^s_{(0,q)}(U)} + \|u\|_{W^t_{(0,q)}(U)}\right), \quad u \in \mathrm{dom}(\Box_q). \tag{2.83}$$

(2.83) is to be understood as a genuine estimate (as opposed to an a priori estimate, where u would be assumed to be in W^{s+2} on V): if the right-hand side is finite, then so is the left-hand side, and the estimate holds. When t is less than s, the last term in (2.83) is weaker than the first: $W^s(U)$ in this case embeds compactly into $W^t(U)$, by Rellich's Lemma. Rewriting (2.83) for $N_q u$ and $t = 0$ and using that $\|N_q u\|_{\mathcal{L}^2_{(0,q)}(U)} \leq \|N_q u\|_{\mathcal{L}^2_{(0,q)}(\Omega)} \lesssim \|u\|_{\mathcal{L}^2_{(0,q)}(\Omega)}$, we find

$$\|N_q u\|_{W^{s+2}_{(0,q)}(V)} \leq C\left(\|u\|_{W^s_{(0,q)}(U)} + \|u\|_{\mathcal{L}^2_{(0,q)}(\Omega)}\right), \quad u \in \mathcal{L}^2_{(0,q)}(\Omega). \tag{2.84}$$

One says that locally, in the interior, N_q gains two derivatives. In particular, if $u \in C^\infty_{(0,q)}(\Omega) \cap \mathcal{L}^2_{(0,q)}(\Omega)$, then so is $N_q u$, by the Sobolev imbedding theorem.

Similarly, we have interior elliptic regularity for $\bar\partial \oplus \vartheta$: if $\bar\partial u$ and ϑu are contained in $W^s_{(0,q+1)}(U)$ and $W^s_{(0,q-1)}(U)$, respectively, then $u \in W^{s+1}_{(0,q)}(V)$ (since then $(\bar\partial\vartheta + \vartheta\bar\partial)u \in W^{s-1}_{(0,q)}(U)$ and $s - 1 \geq -1$ as long as $s \geq 0$). The estimate analogous to (2.83) is therefore, for $s \geq 0$, $t \in \mathbb{R}$:

$$\|u\|_{W^{s+1}_{(0,q)}(V)} \leq C\left(\|\bar\partial u\|_{W^s_{(0,q+1)}(U)} + \|\vartheta u\|_{W^s_{(0,q-1)}(U)} + \|u\|_{W^t_{(0,q)}(U)}\right). \tag{2.85}$$

A second consequence of Lemma 2.11 is the following observation about the normal component of a form in $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$: the normal component is then in $W^1_0(\Omega)$ (rather than just in $\mathcal{L}^2$). For $u \in \mathcal{L}^2_{(0,q)}(\Omega)$, denote by u_{norm} the contraction of the vector field $\sum_{j=1}^n (\partial\rho/\partial z_j)(\partial/\partial\bar z_j)$ with u, i.e., u_{norm} is the $(0, q-1)$ form

$$u_{\mathrm{norm}} := \sum_{|K|=q-1}{}' \left(\sum_{j=1}^n \frac{\partial\rho}{\partial z_j} u_{jK}\right) d\bar z_K. \tag{2.86}$$

Although this is true only up to a smooth factor (which does not vanish near the boundary) when the defining function does not have normalized gradient, we will call u_{norm} 'the' normal component of u. Strictly speaking, u_{norm} as defined depends on the defining function ρ. However, the essential properties, in particular estimate (2.88) below, do not. Accordingly, we will not indicate this dependence unless it is necessary to avoid confusion. Note that the boundary condition (2.9) for membership in $\mathrm{dom}(\bar\partial^*)$ says precisely that

$$u_{\mathrm{norm}} = 0 \quad \text{on } b\Omega. \tag{2.87}$$

One reason for omitting the normalizing factor $1/|\nabla\rho|$ in (2.86) is that $\nabla\rho$ can vanish. However, it is always nonvanishing in a neighborhood of the boundary, and the sublevel sets $\Omega_\varepsilon = \{z \in \mathbb{C}^n \mid \rho(z) < -\varepsilon\}$ are smooth for $\varepsilon \geq 0$ small enough. This gives a well defined decomposition, near the boundary, of a form u into its normal and tangential parts. If $u = \sum_J' u_J d\bar{\omega}_J$ in a special boundary chart, set $u_{\mathrm{Norm}} = \sum_{n\in J}' u_J d\bar{\omega}_J = \sum_{|K|=q-1}' u_{Kn} d\bar{\omega}_K \wedge \bar{\omega}_n$ and $u_{\mathrm{Tan}} = \sum_{n\notin J}' u_J d\bar{\omega}_J$. That is, u_{Norm} collects all the terms that contain $\bar{\omega}_n$, while u_{Tan} collects the terms that do not. This decomposition is independent of the special boundary chart. As a result, u_{Norm} and u_{Tan} can be defined globally (near the boundary) via a suitable partition of unity. Up to a normalizing factor, $u_{\mathrm{Norm}} = u_{\mathrm{norm}} \wedge \bar{\omega}_n$ (because $\sum_{j=1}^n (\partial\rho/\partial\bar{z}_j)(\partial/\partial z_j)$ and ω_n are dual to each other, modulo normalization). Note that in view of (2.87), when u_{Tan} is continued to all of Ω and then restricted to $\Omega_\varepsilon = \{z \in \mathbb{C}^n \mid \rho(z) < -\varepsilon\}$, this restriction is in the domain of $\bar{\partial}^*$ on Ω_ε, for $\varepsilon \geq 0$ small enough.

Often, for example in (2.87) or for questions of membership in Sobolev spaces, it does not matter whether we consider u_{norm} or u_{Norm}.

Lemma 2.12. *Let Ω be a bounded pseudoconvex domain with C^4 boundary. There is a constant C such that if $u \in \mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$, then $u_{\mathrm{norm}} \in W^1_{0,(0,q-1)}(\Omega)$, and*

$$\|u_{\mathrm{norm}}\|_1 \leq C(\|\bar{\partial}u\| + \|\bar{\partial}^* u\|). \tag{2.88}$$

Proof. First assume that $u \in C^1_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$. Then $u_{\mathrm{norm}} = 0$ on $b\Omega$, and for K an increasing $(q-1)$-tuple fixed, we have

$$\|(u_{\mathrm{norm}})_K\|_1 \leq C\|\Delta(u_{\mathrm{norm}})_K\|_{-1}, \tag{2.89}$$

since $\Delta\colon W^1_0(\Omega) \to W^{-1}(\Omega)$ is an isomorphism (see e.g. [122], [215], [299], [294]). Using (2.86) to compute $\Delta(u_{\mathrm{norm}})_K$ shows that this Laplacian is a sum of terms of the form $C^1(\overline{\Omega})$ function times coefficients of u plus $C^1(\overline{\Omega})$ function times first order derivatives of coefficients of u plus $C^1(\overline{\Omega})$ function times Δu_{jK} (we have used here that $\rho \in C^4$). By Lemma 2.11, terms of the last kind can be expressed as combinations of $C^1(\overline{\Omega})$ function times first order derivatives of coefficients of $\bar{\partial}u$ and of $\bar{\partial}^* u$. Because functions in $C^1(\overline{\Omega})$ are bounded multipliers of $W^1_0(\Omega)$, they are bounded multipliers in $W^{-1}(\Omega)$ (by duality). Therefore we obtain from (2.89) that there is a constant such that

$$\|(u_{\mathrm{norm}})_K\|_1 \leq C(\|u\| + \|\bar{\partial}u\| + \|\bar{\partial}^* u\|) \leq C(\|\bar{\partial}u\| + \|\bar{\partial}^* u\|), \tag{2.90}$$

where the last inequality comes from Proposition 2.7. In (2.90), we have made use of the usual convention that a constant may change its value at successive appearances.

When u is only assumed in $\mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$, we use Proposition 2.3 to approximate u in the graph norm by forms in $C^1_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*)$. (2.90) shows that the approximating sequence is Cauchy in $W^1_{0,(0,q-1)}(\Omega)$. Since it converges to u_{norm} in $\mathcal{L}^2_{(0,q-1)}(\Omega)$, passing to the limit shows that $u_{\mathrm{norm}} \in W^1_{0,(0,q-1)}(\Omega)$, and that (2.88) holds. The proof of Lemma 2.12 is complete. □

Lemma 2.12 implies in particular that the normal component of $N_q u$ is in $W^1_{(0,q-1)}(\Omega)$, with the estimate

$$\|(N_q u)_{\text{norm}}\|_1 \lesssim \|\bar{\partial} N_q u\| + \|\bar{\partial}^* N_q u\| \lesssim \|u\|; \tag{2.91}$$

similar estimates hold for the normal components of $\bar{\partial} N_q u$ and $\bar{\partial}^* N_q u$. There is also an estimate dual to (2.91). Let V be the intersection of a small enough neighborhood of $b\Omega$ with Ω so that the decomposition of a from u into its tangential and normal parts is well defined on V. Let U be a neighborhood of $\overline{\Omega \setminus V}$ and choose $\chi_1, \chi_2 \in C_0^\infty(U)$ with χ_1 identically equal to one in a neighborhood of $\overline{\Omega \setminus V}$ and χ_2 identically equal to one in a neighborhood of the support of χ_1. Then we have for $u, v \in \mathcal{L}^2_{(0,q)}(\Omega)$:

$$\begin{aligned}
\left|(N_q u, v)\right| = \left|(u, N_q v)\right| &\leq \left|(u, \chi_2 N_q v)\right| + \left|((1-\chi_2)u, N_q v)\right| \\
&\lesssim \|u\|_{-1}\|\chi_2 N_q v\|_1 + \|(1-\chi_2)u_{\text{Tan}}\|\|N_q v\| \\
&\qquad + \|(1-\chi_1)u_N\|_{-1}\|(1-\chi_2)(N_q v)_N\|_1 \\
&\lesssim (\|u\|_{-1} + \|(1-\chi_2)u_{\text{Tan}}\|)\,\|v\|.
\end{aligned} \tag{2.92}$$

In the last inequality in (2.92) we have used (2.84) and (2.91). (2.92) gives

$$\|N_q u\| \lesssim \|u\|_{-1} + \|(1-\chi_2)u_{\text{Tan}}\| \lesssim \|u\|_{-1} + \|u \wedge \bar{\partial}\rho\| \tag{2.93}$$

(ρ is a defining function for Ω). We have used that $\|(1-\chi_2)u_{\text{Tan}}\| \approx \|(1-\chi_2)(u \wedge \bar{\partial}\rho)\| \lesssim \|u \wedge \bar{\partial}\rho)\|$. (2.93) says that to estimate $\|N_q u\|$, the $\mathcal{L}^2$-norm of only the tangential component is needed, while the normal component enters only with the (-1)-norm. Again, analogous estimates hold for $\bar{\partial} N_q$ and $\bar{\partial}^* N_q$. Combining these with (2.91) yields an improvement of (2.91) that gives the 'correct' gain of two derivatives for N_q 'with respect to the normal components':

$$\|(N_q u)_{\text{norm}}\|_1 \lesssim \|u\|_{-1} + \|u \wedge \bar{\partial}\rho\|. \tag{2.94}$$

A special situation occurs when $q = n$. In this case, a form equals its normal part, and the second term on the right-hand side of (2.94) vanishes. This gives the estimate $\|N_n u\|_1 \lesssim \|u\|_{-1}$. Because $N_n u$ is in $W^1_{(0,n)}(\Omega)$, its trace on the boundary (the boundary value) vanishes in the sense of the elliptic theory of the Dirichlet problem for the Laplacian ([122], [215], [294], [299]). We can therefore invoke Lemma 2.11 to obtain the elliptic estimate

$$\|N_n u\|_{s+2} \leq C_s \|u\|_s. \tag{2.95}$$

Similarly, it is clear from the proof of Lemma 2.12 that when $b\Omega$ is regular enough, there is a version of Lemma 2.12 for higher order Sobolev spaces. For example, when $b\Omega$ is C^∞, we obtain for $s \geq 0$ and $u \in \text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1}) \cap W^s_{(0,q)}(\Omega)$:

$$\|u_{\text{norm}}\|_{s+1} \leq C_s(\|\bar{\partial} u\|_s + \|\bar{\partial}^* u\|_s + \|u\|_s). \tag{2.96}$$

$\|u\|_s$ must be included because we cannot in general infer the last inequality in (2.90) for s-norms when $s > 0$. There are then versions of the estimates above in Sobolev spaces, but one has to assume exact regularity of the operator in question (in $\mathcal{L}^2$, this is not an issue). For example, it is shown in [55], Theorem C, that if $\bar{\partial}^* N_q$ is continuous in Sobolev spaces, then there is an estimate in Sobolev norms corresponding to (2.93).

Remark. There are other useful consequences of Lemma 2.11, based on the observation that the lemma implies that any form $u = \sum'_J u_J d\bar{z}_J \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\vartheta)$ can be written as $u = v + w$, where v has coefficients in $W_0^1(\Omega)$ and w has harmonic coefficients (set $v = \sum'_J v_J d\bar{z}_J$ where v_J is the unique solution in $W_0^1(\Omega)$ of the equation $\Delta v_J = \Delta u_J$; note that $\|\Delta u_J\|_{-1} \lesssim \|\bar{\partial} u\| + \|\vartheta u\|$). The idea of such a decomposition can be traced back at least to [230]. Its usefulness stems from the following two facts about harmonic functions. First, for ε small positive, the ε-norm of a harmonic function g is equivalent to the $\mathcal{L}^2$-norm of g times the boundary distance to the power $-\varepsilon$ ([178]). Second, a harmonic function in $W^s(\Omega)$ has a distributional boundary value which is in $W^{s-1/2}(b\Omega)$, for all $s \in \mathbb{R}$ ([215], Chapter 2, Section 8.1). Applications of the first property to estimates for the $\bar{\partial}$-Neumann operator and related operators can be found for example in [283], [37], [226], [156], and [157]. The second property implies, in view of the above decomposition, that a form in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\vartheta)$ has a distributional boundary value (coefficientwise) in $W^{-1/2}(b\Omega)$; in particular, this applies to the forms $N_q u$, $\bar{\partial} N_q u$, and $\bar{\partial}^* N_q u$. For these ideas, and applications, see [55], [76], and [226].

2.10 Restriction of forms to subdomains

With the $\bar{\partial}$-Neumann operators in hand, we now return briefly to inequalities (2.46) and (2.48) and show that they remain valid without any boundary smoothness assumption. When Ω is only assumed pseudoconvex, it can be exhausted by a sequence of smooth (strictly pseudoconvex) domains $\{\Omega_\nu\}_{\nu=1}^{\infty}$. On these subdomains, (2.46) and (2.48) hold. The reason why these estimates do not pass immediately to Ω is that when $u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ is restricted to Ω_ν, the restriction need not be in the domain of $\bar{\partial}^*$ on Ω_ν (the normal component need not be zero on the boundary of Ω_ν). There is a simple procedure to deal with this difficulty that exploits the $\bar{\partial}$-Neumann operators of the subdomains; this idea comes from [283].

Corollary 2.13. *Let Ω be a bounded pseudoconvex domain, $u = \sum'_J u_J d\bar{z}_J \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*) \subseteq \mathcal{L}^2_{(0,q)}(\Omega)$. Then*

(i) $\partial u_J / \partial \bar{z}_j \in \mathcal{L}^2(\Omega)$, $1 \le j \le n$, *and*

$$\sum_J{}' \sum_{j=1}^{n} \|\partial u_J / \partial \bar{z}_j\|^2 \le \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2; \tag{2.97}$$

(ii) *if $b \in C^2(\overline{\Omega})$, $b \le 0$, then*

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}}\, dV \le \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2. \tag{2.98}$$

Proof. Let $\{\Omega_\nu\}_{\nu=1}^{\infty}$ be an exhaustion of Ω by smooth (strictly) pseudoconvex domains. Given $u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$, define forms on Ω_ν via

$$u_\nu := \bar\partial_\nu^* N_{q+1,\nu}(\bar\partial u|_{\Omega_\nu}) + (N_{q,\nu}\bar\partial_\nu)(\vartheta u|_{\Omega_\nu}), \tag{2.99}$$

where $\cdot|_{\Omega_\nu}$ denotes restriction to Ω_ν and the subscripts indicate that operators are on Ω_ν. The parentheses in the last term in (2.99) indicate that $N_{q,\nu}\bar\partial_\nu$ is to be interpreted as one (bounded) operator on $\mathcal{L}^2$ (since $\vartheta u|_{\Omega_\nu}$ is not necessarily in $\mathrm{dom}(\bar\partial_\nu)$); alternatively, one could use the composition $\bar\partial_\nu N_{q-1,\nu}$ (when $q = 1$ this involves $N_{0,\nu}$; see Remark (i) after the statement of Theorem 2.9 for $N_{0,\nu}$). Now $u_\nu \in \mathrm{dom}(\bar\partial_\nu) \cap \mathrm{dom}(\bar\partial_\nu^*)$, and we obtain from (2.46) (combined with the remark following Lemma 2.6)

$$\begin{aligned}\sum_J{}' \sum_{j=1}^{n} \left\| \frac{\partial (u_\nu)_J}{\partial \bar z_j} \right\|_{\Omega_\nu}^2 &\le \|\bar\partial_\nu \bar\partial_\nu^* N_{q+1,\nu}(\bar\partial u|_{\Omega_\nu})\|_{\Omega_\nu}^2 + \|\bar\partial_\nu^* N_{q,\nu}\bar\partial_\nu(\vartheta u|_{\Omega_\nu})\|_{\Omega_\nu}^2 \\ &\le \|\bar\partial u|_{\Omega_\nu}\|_{\Omega_\nu}^2 + \|\vartheta u|_{\Omega_\nu}\|_{\Omega_\nu}^2 \le \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2.\end{aligned} \tag{2.100}$$

Here we have used that $\bar\partial_\nu \bar\partial_\nu^* N_{q+1,\nu}$ and $\bar\partial_\nu^* N_{q,\nu}\bar\partial_\nu$ are orthogonal projections. In view of (2.69), we similarly have

$$\|u_\nu\|_{\Omega_\nu}^2 \le \frac{D^2 e}{q}(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2), \tag{2.101}$$

where D is the diameter of Ω. Consequently, when the forms u_ν are continued by zero on $\Omega \setminus \Omega_\nu$, a suitable subsequence will have a weak limit in $\mathcal{L}^2_{(0,q)}(\Omega)$, say w. By passing to a further subsequence, we may assume, by (2.100), that the functions $\partial(u_\nu)_J/\partial \bar z_j$, continued by zero outside Ω, converge weakly in $\mathcal{L}^2(\Omega)$, for all J and j. Denote these weak limits by $w_{j,J}$. The proof of (i) will be complete if we can show that $w = u$, because this implies that $w_{j,J} = \partial u_J/\partial \bar z_j$ (as distributions); this in turn gives the desired estimate (2.97) (in view of (2.100)).

We first note that $w \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$. For a smooth compactly supported $(0, q+1)$-form α, the distributional pairing $(\bar\partial w, \alpha)$ equals

$$(w, \bar\partial^* \alpha) = \lim_{\nu\to\infty} (u_\nu, \bar\partial^* \alpha)_{\Omega_\nu} = \lim_{\nu\to\infty} (\bar\partial u_\nu, \alpha)_{\Omega_\nu}.$$

But $\bar\partial u_\nu = \bar\partial u|_{\Omega_\nu}$ on Ω_ν (see (2.99)), so we obtain $= \lim_{\nu\to\infty}(\bar\partial u, \alpha)_{\Omega_\nu} = (\bar\partial u, \alpha)$, whence $\bar\partial w = \bar\partial u$. When $\beta \in \mathrm{dom}(\bar\partial)$, $|(u_\nu, \bar\partial\beta)_{\Omega_\nu}| = \left|\left(\bar\partial_\nu^* N_{q,\nu}\bar\partial(\vartheta u|_{\Omega_\nu}), \beta\right)_{\Omega_\nu}\right| \le \|\vartheta u\|_{\Omega_\nu}\|\beta\|_{\Omega_\nu} \le \|\vartheta u\|\|\beta\|$. Therefore, $|(w, \bar\partial\beta)| \le \|\vartheta u\|\|\beta\|$, and $w \in \mathrm{dom}(\bar\partial^*)$. Computing $\bar\partial^* w$ as the (distributional) limit of ϑu_ν (extended by zero outside ω_ν), we find that it equals the limit of $\bar\partial_\nu^* N_{q,\nu}\bar\partial_\nu(\vartheta u|_{\Omega_\nu}) = \vartheta u|_{\Omega_\nu} - P_{q-1,\nu}(\vartheta u|_{\Omega_\nu})$. (Recall that $P_{q-1,\nu}$ is the orthogonal projection of $\mathcal{L}^2_{(0,q-1)}(\Omega_\nu)$ onto $\ker(\bar\partial)$.) The weak limit of $P_{q-1,\nu}(\vartheta u|_{\Omega_\nu})$ is $\bar\partial$ closed, and its difference to ϑu is orthogonal to $\ker(\bar\partial)$. Consequently, this limit equals $P_{q-1}\vartheta u$. As a result, $\bar\partial^* w = \vartheta u - P_{q-1}\vartheta u =$

$\bar\partial^* u - P_{q-1}\bar\partial^* u = \bar\partial^* u$. Since also $\bar\partial u = \bar\partial w$, it follows that $u = w$. This completes the proof of (i).

To see (ii), fix a smooth relatively compact subdomain $\widehat{\Omega}$ of Ω, and $\varphi \in C_0^\infty(\Omega)$ with $\varphi \equiv 1$ in a neighborhood of the closure of $\widehat{\Omega}$. By the interior elliptic regularity of $\bar\partial \oplus \vartheta$ ((2.85)), we have the estimate

$$\begin{aligned}\|\varphi u_\nu\|^2_{\Omega_\nu,1} &\lesssim \|\bar\partial_\nu(\varphi u_\nu)\|^2_{\Omega_\nu} + \|\bar\partial^*_\nu(\varphi u_\nu)\|^2_{\Omega_\nu}\\ &\lesssim \|u\|^2_{\Omega_\nu} + \|\overline{\partial_\nu} u_\nu\|^2_{\Omega_\nu} + \|\bar\partial^*_\nu u_\nu\|^2_{\Omega_\nu},\end{aligned} \tag{2.102}$$

with constants independent of ν, ν big enough ($\widehat{\Omega}$ and φ fixed). As in (2.100), the right-hand side of (2.102) is bounded independently of ν. Consequently, a suitable subsequence of $\{u_\nu\}_{\nu=1}^\infty$ converges in $\mathcal{L}^2_{(0,q)}(\widehat{\Omega})$ ($W^1(\widehat{\Omega})$ embeds compactly into $\mathcal{L}^2(\widehat{\Omega})$); from what was shown in the proof of (i), it is clear that this limit is (the restriction to $\widehat{\Omega}$ of) u. Applying (2.98) to u_ν on Ω_ν (this is Lemma 2.6) and again using that $\|\overline{\partial_\nu} u_\nu\|^2_{\Omega_\nu} + \|\bar\partial^*_\nu u_\nu\|^2_{\Omega_\nu} \le \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$ (as in (2.100)) now gives (2.98) with the integration on the left-hand side over $\widehat{\Omega}$. Since $\widehat{\Omega}$ was arbitrary, this implies (ii). □

Remarks. (i) In applications of (2.98), the form $\sum'_{|K|=q-1}\sum_{j,k=1}^n e^b \frac{\partial^2 b}{\partial z_j \partial \bar z_k} u_{jK}\overline{u_{kK}}$ is typically positive semidefinite. In this case, it suffices that $b \in C^2(\Omega)$ (as opposed to $C^2(\overline{\Omega})$): in the very last step of the proof, convergence of the integrals over $\widehat{\Omega}$ to the integral over Ω follows from Fatou's lemma when the integrands are positive, without any boundary regularity of b.

(ii) Note that in the above arguments, the various weak limits are independent of the particular subsequence as $\nu \to \infty$. This implies that the weak limits exist as $\nu \to \infty$: it is not necessary to pass to suitable subsequences. In particular, if $u \in \mathcal{L}^2_{(0,q)}(\Omega)$, then $P_{q,\nu}(u|_{\Omega_\nu})$ (continued by zero outside Ω_ν) converges weakly to $P_q u$ in $\mathcal{L}^2_{(0,q)}(\Omega)$. Variants of these arguments also yield the weak convergence of $N_{q,\nu}u$ (continued by zero outside Ω_ν) to $N_q u$. These ideas occur for example in [44], proof of Theorem 2, [283], proof of Theorem 1, and [225], proof of Theorem 1.2 .

2.11 Solvability of $\bar\partial$ in $C^\infty_{(0,q)}(\Omega)$

So far we have used Proposition 2.4 only in the simplified form with $\varphi \equiv 0$. We now give a first illustration of the flexibility afforded by the exponential factor. Assume Ω is a bounded pseudoconvex domain, and consider a form $u \in C^\infty_{(0,q)}(\Omega)$, with $\bar\partial u = 0$. Write $\Omega = \cup\Omega_m$, where $\{\Omega_m\}_{m\in\mathbb{N}}$ is an exhaustion of Ω by (C^2) strictly pseudoconvex domains. Then $u \in \mathcal{L}^2_{(0,q)}(\Omega_m)\cap\ker(\bar\partial)$ for all m, and consequently there are $(q-1)$-forms $f_m \in \mathcal{L}^2_{(0,q-1)}(\Omega_m)$ with $\bar\partial f_m = u$ on Ω_m and $\|f_m\|^2_{\Omega_m} \le (D^2e/q)\|u\|^2_{\Omega_m}$, by Theorem 2.9 and Corollary 2.10 (where D is the diameter of Ω; it dominates the diameter of Ω_m, for all m). Unfortunately, this does not produce a solution of $\bar\partial f = u$ on Ω (say by taking weak limits) because $\|u\|_{\Omega_m}$ may be unbounded as $m \to \infty$. However, for suitably chosen φ, $u \in \mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi})$, where $\mathcal{L}^2(\Omega, e^{-\varphi})$ denotes the space of functions square integrable with respect to the weight $e^{-\varphi}$; and the full version of Proposition 2.4 can be exploited to prove solvability of $\bar{\partial}$ in the $C^\infty(\Omega)$ category ([172], Corollary 4.2.6, see already [170], Section 2.2).

Theorem 2.14. *Assume Ω is a bounded pseudoconvex domain, and $u \in C^\infty_{(0,q)}(\Omega)$, $1 \le q \le n$, with $\bar{\partial} u = 0$. Then there exists $f \in C^\infty_{(0,q-1)}(\Omega)$ such that $\bar{\partial} f = u$.*

Proof. Fix $u = u_0$ as given in the theorem. We first show that there exists a smooth plurisubharmonic function φ on Ω that grows fast enough towards the boundary that $u_0 \in \mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi})$. We use the following fact from the classical theory of several complex variables: there exists a smooth plurisubharmonic exhaustion function h for Ω ([207], Theorem 3.3.5, [259], paragraph II.5). We look for φ in the form $\varphi(z) = g(h(z))$ for a suitable smooth function g. Plurisubharmonicity of φ means that

$$\begin{aligned}\sum_{j,k} &\frac{\partial^2}{\partial z_j \partial \bar{z}_k} g(h(z)) w_j \overline{w_k} \\ &= g''(h(z)) \left| \sum_j \frac{\partial h}{\partial z_j}(z) w_j \right|^2 + g'(h(z)) \sum_{j,k} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}(z) w_j \overline{w_k} \ge 0,\end{aligned} \tag{2.103}$$

where $z \in \Omega$, $w \in \mathbb{C}^n$. Since h is plurisubharmonic, any g such that g' is positive and increasing will do. u will be in $\mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi})$ if, say, $\varphi(z) = g(h(z)) \ge 2\log(|u(z)|+1)$ near the boundary (since Ω is bounded). Note that the exhaustion function h is bounded from below; without loss of generality, we may assume that $h \ge 0$ on Ω. Set $\sigma_m := \min_{z \in \overline{\Omega_m} \setminus \Omega_{m-1}} h(z)$ (where $\Omega_0 := \emptyset$). Because h is an exhaustion function, $\sigma_m \to \infty$ as $m \to \infty$. Next, set $\mu_m := \max_{z \in \overline{\Omega_m}} 2\log(|u(z)| + 1)$. It now suffices to choose a smooth increasing convex function g on $[0, \infty)$ such that $g(\sigma_m) \ge \mu_m$ for $m = 1, 2, \ldots$. This can be done starting from the piecewise constant increasing function $\hat{g}(x) := \max\{\mu_j | \sigma_j \in [0, x]\} = \mu_{j_0}$, where $j_0 = \max\{j | \sigma_j \in [0, x]\}$. Then $g(h(z)) \ge g(\sigma_m) \ge \mu_m \ge 2\log(|u(z)| + 1)$, where m is the smallest integer such that $z \in \Omega_m$.

Fix such a φ. Also fix m for the time being. Let $b \in C^2(\overline{\Omega_m})$, $b \le 0$. Then

$$\|\bar{\partial} u\|^2_\varphi + \|\bar{\partial}^*_\varphi u\|^2_\varphi \ge {\sum_{|K|=q-1}}' \sum_{j,k=1}^n \int_{\Omega_m} e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}} e^{-\varphi} \, dV, \tag{2.104}$$

for all $u \in \operatorname{dom}(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*_\varphi)$. Here the $\mathcal{L}^2$ spaces are the spaces $\mathcal{L}^2_{(0,q)}(\Omega_m, e^{-\varphi})$. (2.104) follows from Proposition 2.4 with $a = 1 - e^b$, and φ as chosen, in exactly the same manner that (2.48) was derived. Note that the density result from Proposition 2.3 used in the proof remains true for the weighted operators (see the remark after (2.23)). (2.104) implies the analogue of (2.49), again upon choosing $b(z) = -1 + |z - P|^2/D_m$ (also noting that the diameter D of Ω dominates the diameter D_m of Ω_m):

$$\|u\|^2_\varphi \le \frac{D^2 e}{q} (\|\bar{\partial} u\|^2_\varphi + \|\bar{\partial}^*_\varphi u\|^2_\varphi), \quad u \in \operatorname{dom}(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*_\varphi). \tag{2.105}$$

The analysis following Proposition 2.7 now applies essentially verbatim, yielding weighted operators $\Box_{\varphi,q}$ and $N_{\varphi,q}$, on Ω_m, with the (weighted versions of the) properties in Theorem 2.9 and Corollary 2.10. (Recall that Ω_m has C^2 boundary, so the argument needed in the proof of Proposition 2.7 to accommodate domains with possibly irregular boundaries is not needed at this point.) In particular, we have the weighted canonical solution operator $\bar\partial^*_\varphi N_{\varphi,q}$, and

$$\begin{aligned}\bar\partial(\bar\partial^*_\varphi N_{\varphi,q}u_0) &= u_0 \quad \text{on } \Omega_m,\\ \text{with } \|\bar\partial^*_\varphi N_{\varphi,q}u_0\|^2_\varphi &\le \frac{D^2e}{q}\|u_0\|^2_{\varphi,\Omega_m} \le \frac{D^2e}{q}\|u_0\|^2_\varphi.\end{aligned} \tag{2.106}$$

Of course, we have used that $u_0 \in \mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi})$, so that $u_0|_{\Omega_m} \in \mathcal{L}^2_{(0,q)}(\Omega_m, e^{-\varphi})$. To shorten notation, set $f_m := \bar\partial^*_\varphi N_{\varphi,q}u_0$.

We claim that $f_m \in C^\infty_{(0,q)}(\Omega_m)$. This is a consequence of the interior elliptic regularity of $\bar\partial \oplus \vartheta$ as given in (2.85). Fix a relatively compact open subset of Ω_m. By (2.23), we have on Ω_m

$$\bar\partial^* f_m = \bar\partial^*_\varphi f_m - {\sum_{|K|=q-2}}' \left(\sum_{j=1}^n \frac{\partial\varphi}{\partial z_j}(f_m)_{jK}\right)d\bar z_K. \tag{2.107}$$

When $q = 1$, the sum over K is to be interpreted as 0. Note that $f_m \in \mathcal{L}^2_{(0,q-1)}(\Omega_m)$. Therefore, since $\bar\partial^*_\varphi f_m = 0$, $\bar\partial^* f_m$ is square integrable on Ω_m. Of course, so is $\bar\partial f_m = u_0$ (u_0 is smooth on Ω). Consequently, (2.85) gives that $f_m|_V \in W^1_{(0,q-1)}(V)$. V was an arbitrary relatively compact subdomain of Ω_m. Therefore, we can repeat the above argument for open sets V_1, V_2 with $V_1 \subset\subset V_2 \subset\subset \Omega_m$. (2.107) now gives that $\bar\partial^* f_m$ is in W^1 on V_2, and since $\bar\partial f_m = u_0$ also is, (2.85) gives that actually f_m is in W^2 on V_1. So f_m is in W^2 on every relatively compact subdomain of Ω_m. Because φ is smooth in Ω, we can continue this procedure to obtain inductively that f_m is in W^k for all $k \in \mathbb{N}$ on every relatively compact subdomain of Ω_m. Now the Sobolev Lemma implies that $f_m \in C^\infty_{(0,q-1)}(\Omega_m)$.

We now produce a solution on Ω from the f_m. Define $\widetilde{f_m} \in \mathcal{L}^2_{(0,q-1)}(\Omega, e^{-\varphi})$ by setting it equal to f_m on Ω_m and to 0 on $\Omega \setminus \Omega_m$. By (2.106), $\{\widetilde{f_m}\}$ is a bounded sequence in $\mathcal{L}^2_{(0,q-1)}(\Omega)$. Let f be the limit of a weakly convergent subsequence. One can check in the above inductive argument that for a fixed relatively compact subdomain V of Ω, and (fixed) $k \in \mathbb{N}$, $\|\widetilde{f_m}\|_{W^k_{(0,q-1)}(V)} = \|f_m\|_{W^k_{(0,q-1)}(V)}$ is bounded independently of m for m big enough (i.e., $V \subset\subset \Omega_m$). Consequently, a further subsequence of $\{\widetilde{f_m}\}$ converges weakly in $W^k_{(0,q-1)}(V)$. This weak limit must equal the restriction of f to V. Since k and V were arbitrary, it follows that $f \in C^\infty_{(0,q-1)}(\Omega)$. Moreover, weak convergence in $\mathcal{L}^2(\Omega, e^{-\varphi})$ implies convergence as distributions, whence $\bar\partial f = u_0$ (first as distributions, but since both f and u_0 are smooth, also as smooth forms). This completes the proof of Theorem 2.14. $\square$

Remarks. (i) For domains of holomorphy, solvability in $C^\infty(\Omega)$ (so $q = 1$) goes back implicitly to Oka's paper [245], where it is shown that the first Cousin problem is always solvable on a bounded domain of holomorphy in $\mathbb{C}^n$. But modulo solving $\bar\partial$ *locally*, say on a polydisc (the Dolbeault–Grothendieck lemma), which is elementary, the two problems are equivalent (see for example [97], Chapter 6, [207], Chapter 6). For all q (still on domains of holomorphy), this existence theorem was first obtained with the sheaf theoretic methods of the early fifties. Indeed, Cartan's Theorem B ([61], [267], [154], p. 96, [97], p. 281, [207], Theorem 7.1.7 (warning: here, the theorem is stated for pseudoconvex domains)) gives that in this situation $H^q(\Omega, \mathcal{O})$ vanishes for $1 \le q \le n-1$, where $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions, and the Dolbeault isomorphism ([118], [119], [97], Corollary 6.40, [207], Theorem 6.3.1) equates this sheaf cohomology group with the q-th $\bar\partial$-cohomology. The solution of the Levi problem (the fact that pseudoconvex domains are domains of holomorphy), achieved in the early fifties also (see Remarks (ii) and (iii) following the proof of Theorem 2.16 below) then implies solvability on pseudoconvex domains. For additional information on the use of sheaf theoretic (i.e., algebraic) methods in complex analysis, see [172], [154], [207], [97], [150].

(ii) It is a consequence of the 'classical' Cartan–Thullen convexity theory ([62]) that domains of holomorphy are pseudoconvex (see [259], Chapter II, Section 3, in particular Theorem 3.24), so one can obtain the results in the previous remark from Theorem 2.14. More importantly, the fact that the $\mathcal{L}^2$-methods of this chapter work on pseudoconvex domains directly also leads to a solution of the Levi problem: once Theorem 2.14 is established, it is fairly easy to prove that pseudoconvex domains are domains of holomorphy (compare again Remarks (ii) and (iii) following the proof of Theorem 2.16 below).

2.12 Non-plurisubharmonic weights

There is considerable additional flexibility in Proposition 2.4. For a thorough discussion of the relevant ideas, we refer the reader to the surveys [36], [223]. In the following, we provide a brief discussion (following, more or less, [243], [223]). Returning to the general setup in Proposition 2.4, we obtain (pseudoconvexity implies as before that the boundary term in (2.24) is nonnegative)

$$\|\sqrt{a}\bar\partial u\|_\varphi^2 + \|\sqrt{a}\bar\partial_\varphi^* u\|_\varphi^2 \ge \int_\Omega {\sum_K}' \sum_{j,k=1}^n \left(a\frac{\partial^2\varphi}{\partial z_j \partial \bar z_k} - \frac{\partial^2 a}{\partial z_j \partial \bar z_k}\right) u_{jK}\overline{u_{kK}} e^{-\varphi} + 2\,\mathrm{Re}\left({\sum_K}' \sum_{j=1}^n u_{jK}\frac{\partial a}{\partial z_j}, \bar\partial_\varphi^* u\right)_\varphi . \tag{2.108}$$

Let b again be a C^2-function on $\overline{\Omega}$ (but not necessarily nonpositive as in (2.47)). As in (2.47), inserting a factor $e^{-b/2}$ on the left-hand side of the inner product in the second line of (2.108), and a corresponding factor $e^{b/2}$ on the right-hand side, and then invoking the Cauchy–Schwarz inequality gives

$$\int_\Omega \sum_K{}' \sum_{j,k=1}^n \left(a\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k} - \frac{\partial^2 a}{\partial z_j\partial\bar{z}_k} - e^{-b}\frac{\partial a}{\partial z_j}\frac{\partial a}{\partial\bar{z}_k}\right) u_{jK}\overline{u_{kK}}e^{-\varphi}$$
$$\le \|\sqrt{a}\,\bar{\partial}u\|_\varphi^2 + \left\|\sqrt{a+e^b}\,\bar{\partial}_\varphi^* u\right\|_\varphi^2 = \|\sqrt{a}\,\bar{\partial}u\|_\varphi^2 + \left\|\left(\bar{\partial}\sqrt{a+e^b}\right)_\varphi^* u\right\|_\varphi^2. \tag{2.109}$$

u was assumed to be in $C^1_{(0,1)}(\overline{\Omega})$, but (2.109) persists for $u \in \mathrm{dom}(\bar{\partial})\cap\mathrm{dom}(\bar{\partial}^*)$, again by density in the graph norm (Proposition 2.3). (2.109) gives useful estimates when the Hermitian form in the integrand on the left-hand side is strictly positive definite. When $q = 1$, this amounts to the matrix $G_{jk}(z) = \left(a\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k} - \frac{\partial^2 a}{\partial z_j\partial\bar{z}_k} - e^{-b}\frac{\partial a}{\partial z_j}\frac{\partial a}{\partial\bar{z}_k}\right)_{jk}(z)$ being uniformly strictly positive definite on $\overline{\Omega}$ (see Lemma 4.7 when $q > 1$). In this case, so is the matrix $\overline{G(z)}$, which therefore has a Hermitian square root $(\overline{G})^{1/2}(z)$ which is as smooth as G. If $u = \sum_J{}' u_J(z)d\bar{z}_J$, we set $(\overline{G})^{1/2}u(z) = \sum_K{}'\sum_{j,k}((\overline{G})^{1/2})_{jk}(z)u_{kK}(z)d\bar{z}_{jK}$. If $c > 0$ denotes a lower bound for the eigenvalues of G (equivalently: of $\overline{G}$), we can rewrite (2.109) as

$$\begin{aligned} qc\|u\|_\varphi^2 \le q\|(\overline{G})^{1/2}u\|_\varphi^2 &= \int_\Omega \sum_K{}'\sum_j\left|\sum_k((\overline{G})^{1/2})_{jk}u_{kK}\right|^2 e^{-\varphi} \\ &= \int_\Omega \sum_K{}'\sum_{j,k} G_{jk}u_{jK}\overline{u_{kK}}e^{-\varphi} \\ &\le \|\sqrt{a}\,\bar{\partial}u\|_\varphi^2 + \left\|\left(\bar{\partial}\sqrt{a+e^b}\right)_\varphi^* u\right\|_\varphi^2; \end{aligned} \tag{2.110}$$

we have used that G is Hermitian in the second equality. One can now replace the form Q in (2.58) with the right-hand side of (2.110) and define a $\Box$ operator. (2.110) then implies that this operator can be inverted, yielding an associated 'Neumann' operator (essentially equivalently, the framework in Section 1.1 of [170] also applies; note that $(\sqrt{a}\bar{\partial})\circ(\bar{\partial}\sqrt{a+e^b}) = 0$). We only derive estimates for solutions to $\bar{\partial}$, arguing directly from (2.110).

Proposition 2.15. *Let Ω be a bounded pseudoconvex domain, a, b and φ in $C^2(\Omega)$, $a \ge 0$, such that the resulting matrix G is strictly positive definite on Ω. Let $0 \le q \le n-1$. If $u \in \ker(\bar{\partial})$ and $(\overline{G})^{-1/2}u \in \mathcal{L}^2_{(0,q+1)}(\Omega, e^{-\varphi})$, there is a solution f to $\bar{\partial}f = u$ which satisfies the estimate*

$$\left\|\frac{f}{\sqrt{a+e^b}}\right\|_\varphi^2 \le \frac{1}{q^2}\int_\Omega \sum_K{}'\sum_{j,k=1}^n (G^{-1})_{jk}u_{jK}\overline{u_{kK}}e^{-\varphi}. \tag{2.111}$$

Proof. We first assume that Ω has C^2 boundary and that a, b, and φ are in $C^2(\overline{\Omega})$. The arguments are standard and by now familiar. They follow those from (2.49) through the proof of Proposition 2.7. Consider the operator $T = \bar\partial\sqrt{a+e^b}\colon \mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi}) \to \mathcal{L}^2_{(0,q+1)}(\Omega, e^{-\varphi})$. Using (2.110) in place of (2.49), we obtain that the range of T equals the kernel of $\sqrt{a}\bar\partial$, which equals $\ker(\bar\partial)$. To estimate the norm of the solution $\tilde f$ to $T\tilde f = u$ that is orthogonal to the kernel of T, we argue as in (2.51) and pair with an element in the range of T^*_φ:

$$\begin{aligned}|(\tilde f, T^*_\varphi v)_\varphi|^2 = |(u,v)_\varphi|^2 &= \left|\left((\bar G)^{-1/2}u, (\bar G)^{1/2}v\right)_\varphi\right|^2 \\ &\le \left\|(\bar G)^{-1/2}u\right\|^2_\varphi \left\|(\bar G)^{1/2}v\right\|^2_\varphi \le \frac{1}{q}\left\|(\bar G)^{-1/2}u\right\|^2_\varphi \left\|T^*_\varphi v\right\|^2_\varphi;\end{aligned} \tag{2.112}$$

the last inequality is (2.110) applied to v. It follows that $\|\tilde f\|^2_\varphi$ is dominated by $(1/q)\|(\bar G)^{-1/2}u\|^2_\varphi$, which equals the right-hand side of (2.111). Now $f := \sqrt{a+e^b}\,\tilde f$ provides a solution to $\bar\partial f = u$ that satisfies (2.111).

The general case can be handled by exhaustion of Ω by C^2 (strictly) pseudoconvex domains Ω_m, as in the proofs of Proposition 2.7, Corollary 2.13 (see also Remark (i) following that proof), and Theorem 2.14. We leave the details to the reader. □

Remarks. (i) Note that the weight function φ in Proposition 2.15 is not required to be plurisubharmonic. Also, we may replace φ by $\varphi + \lambda$ in (2.111), without changing c, where λ is any plurisubharmonic function: doing so will only 'increase' G (by the Hessian of λ), hence the same constant c in (2.110) will still work. Likewise, the same matrix G will still work in (2.111). Then Proposition 2.15 corresponds to Theorem B in [223] (the roles of φ and λ are interchanged).

(ii) We will use Proposition 2.15 to prove the Ohsawa–Takegoshi extension theorem (Theorem 2.17 below). More precisely, we will use the following variant. Suppose we are in the situation where all the functions involved (a, b, and φ) are in $C^2(\bar\Omega)$, as in the first part of the proof of Proposition 2.15. We only require that $G(z)$ is positive semidefinite on $\bar\Omega$. Then $(\bar G)^{1/2}$ is still well defined (but in general only continuous), and (2.110) still applies. The fact that G may no longer be invertible can be compensated for by the following assumption on u. View the pointwise defined operator $u(z) \to \bar G u(z)$ as an operator on $\mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi})$; it is selfadjoint. The assumption on u is now that u is contained in a (possibly one dimensional) subspace H_u of $\mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi})$ that is invariant under $\bar G$ and on which $\bar G$ is strictly positive (as a selfadjoint operator; in particular, the restriction $\bar G|_{H_u}$ is invertible). Then the conclusion of Proposition 2.15 holds, but with (2.111) modified to

$$\left\|\frac{f}{\sqrt{a+e^b}}\right\|^2_\varphi \le \frac{1}{q}\left((\bar G|_{H_u})^{-1}u, u\right)_\varphi = \frac{1}{q}\left\|(\bar G|_{H_u})^{-1/2}u\right\|^2_\varphi. \tag{2.113}$$

The proof is along the lines of the proof of Proposition 2.15, but it is convenient to slightly reformulate it. Let $v \in \mathrm{dom}(T^*_\varphi)$. Denote by P_{H_u} the orthogonal projection of $\mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi})$ onto H_u. Then

$$\begin{aligned} |(u,v)_\varphi|^2 = |(u, P_{H_u}v)_\varphi|^2 &= \left|\left((\bar G|_{H_u})^{-1/2}u, (\bar G|_{H_u})^{1/2}P_{H_u}v\right)_\varphi\right|^2 \\ &\le \left\|(\bar G|_{H_u})^{-1/2}u\right\|^2_\varphi \left\|(\bar G|_{H_u})^{1/2}P_{H_u}v\right\|^2_\varphi \\ &\le \left\|(\bar G|_{H_u})^{-1/2}u\right\|^2_\varphi \left\|(\bar G)^{1/2}v\right\|^2_\varphi \\ &\le \frac{1}{q}\left\|(\bar G|_{H_u})^{-1/2}u\right\|^2_\varphi \left\|T^*_\varphi v\right\|^2_\varphi, \end{aligned} \tag{2.114}$$

where the last estimate is a consequence of (2.110), applied to v. We have also used that $\left\|(\bar G|_{H_u})^{1/2}P_{H_u}v\right\|^2_\varphi \le \left\|(\bar G)^{1/2}v\right\|^2_\varphi$, which follows because $(H_u)^\perp$ is also invariant under $\bar G$, hence under $(\bar G)^{1/2}$ (in particular, $(G|_{H_u})^{1/2} = (G^{1/2})|_{H_u}$), and $\bar G$ is positive semidefinite. The estimate $|(u,v)_\varphi| \le C\|T^*_\varphi v\|_\varphi$ implies that there is $\tilde f$ with $T\tilde f = u$ and $\|\tilde f\|^2_\varphi \le C$. Indeed, the estimate says that $T^*_\varphi v \to (u,v)_\varphi$ is a bounded linear functional on $\operatorname{Im} T^*_\varphi$. It extends to the closure $\overline{\operatorname{Im}(T^*_\varphi)}$ by continuity, and by the Riesz theorem, there is $\tilde f \in \overline{\operatorname{Im}(T^*_\varphi)}$ such that $(u,v)_\varphi = (\tilde f, T^*_\varphi v)_\varphi$. This says that $\tilde f \in \operatorname{dom}(T^{**}_\varphi) = \operatorname{dom}(T)$, that $T\tilde f = T^{**}_\varphi \tilde f = u$, and that $\|\tilde f\|_\varphi \le C$. Here, $C = (1/q)\|(\bar G|_{H_u})^{-1/2}u\|^2$ (from (2.114)). We will use this version in the proof of Theorem 2.17.

2.13 $\bar\partial$-techniques, the Ohsawa–Takegoshi extension theorem

We close this chapter with an illustration of what are usually referred to as '$\bar\partial$-techniques'. Roughly speaking, this means the following. Suppose one looks for a holomorphic function (or some other 'holomorphic' object, such as a $\bar\partial$-closed form) with certain properties. Often, it is easy to find a C^∞ function with the desired property. Say g is such a function. Now the idea is to modify g in such a way that the result is holomorphic and so that the modification preserves the desired property. The requirement that the modification be holomorphic leads to a $\bar\partial$-equation for a function involved in the modification. Results such as Theorem 2.14, Corollary 2.10, or Proposition 2.15, then allow to solve this equation in the appropriate category, and the problem is solved. This simple strategy, with modifications, turns out to be surprisingly powerful. (We will give another example at the end of Chapter 3 (Theorem 3.7).) It is noteworthy that while these techniques originated with several complex variables, they are useful in the function theory of a single complex variable as well (see for example [172], (proof of) Theorem 1.4.5).

We consider the problem of extending holomorphic functions from the intersection of a pseudoconvex domain with a complex hyperplane to the whole domain (as holomorphic functions).

Theorem 2.16. *Let Ω be a bounded pseudoconvex domain, H a complex hyperplane, and f a holomorphic function in $\Omega \cap H$. Then there exists a holomorphic function F on Ω with $F|_H = f$.*

Proof. We first construct a smooth extension. Assume that coordinates are chosen so that $H = \{z \in \mathbb{C}^n \mid z_n = 0\}$. Denote by π_n the projection onto H: $\pi_n(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, 0)$. $H \cap \Omega$ and $\Omega \setminus \pi_n^{-1}(H \cap \Omega)$ are both relatively closed in Ω. Consequently, there exists a smooth function φ in Ω with values in $[0, 1]$ that is one on $H \cap \Omega$ and zero on $\Omega \setminus \pi_n^{-1}(H \cap \Omega)$ (by the C^∞ version of the Tietze–Urysohn Lemma). Composing this function with a smooth function from $[0, 1]$ to $[0, 1]$ that is zero on $[0, 1/4]$ and one on $[3/4, 1]$ shows that we may actually assume that φ vanishes in a (relative) neighborhood of $\Omega \setminus \pi_n^{-1}(H \cap \Omega)$ and equals one in a neighborhood of $H \cap \Omega$. This guarantees that the function g, defined by

$$g(z) = \begin{cases} \varphi(z) f(\pi_n(z)), & z \in \Omega \cap \pi_n^{-1}(H \cap \Omega), \\ g(z) = 0, & z \in \Omega \setminus \pi_n^{-1}(H \cap \Omega), \end{cases} \tag{2.115}$$

is smooth on Ω. Of course, g extends f. Now we modify g. In order to make sure that the modified function still agrees with f on $H \cap \Omega$, we take it in the form $g + z_n h$. Holomorphy of this modified function is equivalent to

$$\bar\partial(g + z_n h) = 0, \quad \text{or} \quad \bar\partial h = -\frac{\bar\partial g}{z_n}. \tag{2.116}$$

Because $f(\pi_n(z))$ is holomorphic where $\varphi > 0$, and because φ equals one in a relative neighborhood of $H \cap \Omega$ in Ω, $\bar\partial g$ vanishes in this neighborhood, so that the right-hand side of (2.116) is smooth and $\bar\partial$-closed on Ω (strictly speaking: extends by zero into $\{z_n = 0\}$ as a smooth $\bar\partial$-closed form). By Theorem 2.14, there is $h \in C^\infty(\Omega)$ that solves (2.116). But then $F := g + z_n h$ is smooth on Ω, with $\bar\partial F = 0$. So F is holomorphic. And $F|_H = g|_H = f$. □

Remarks. (i) The ideas in the proof of Theorem 2.16 generalize to give extension of holomorphic functions from a submanifold M of Ω that is the zero set of a holomorphic function with nonvanishing gradient; see for example [153], Theorem E7. For a sheaf theoretic approach, on domains of holomorphy, see for example [207], Theorem 7.2.8.

(ii) There is more to Theorem 2.16 than meets the eye. The theorem also works for extension of $\bar\partial$-closed forms (with the same proof). This allows to prove that a domain Ω with the property that for every $u \in C^\infty_{(0,q)}(\Omega) \cap \ker(\bar\partial)$ there is $f \in C^\infty_{(0,q-1)}(\Omega)$ with $\bar\partial f = u$, $1 \le q \le n$, (as in Theorem 2.14) is a so called domain of holomorphy. This means roughly speaking that there exist holomorphic functions that cannot be holomorphically continued past any boundary point (see for example [81], [172], [207], [259] for the precise definition). The argument is by induction on the dimension. When $n = 1$, every domain is a domain of holomorphy. In the induction step, one considers intersections of the domain Ω with a hyperplane H through a boundary point P. By Theorem 2.16, this intersection also has the property that $\bar\partial$ is always solvable in the $C^\infty(\Omega)$ category (and at all form levels): extend a $\bar\partial$-closed $(0, q)$-form to Ω as a $\bar\partial$-closed $(0, q)$-form, solve $\bar\partial$ on Ω, and restrict the resulting $(0, q-1)$-form to $H \cap \Omega$.

(Note that this extension involves solving $\bar{\partial}$ for a $(0, q+1)$-form on Ω.) By induction assumption, there is a holomorphic function on $H \cap \Omega$ with bad boundary behavior at P. Extend this function to a holomorphic function on Ω (again by Theorem 2.16); the extension still has bad boundary behavior at P. Since P was arbitrary, classical methods of function theory show that Ω is indeed a domain of holomorphy. There are some difficulties with this argument that need to be taken care of. For example, it need not be the case that for every boundary point there is a hyperplane H so that the point is also in the boundary of $H \cap \Omega$. There are various ways to address these issues; we refer the reader to [207], Chapter 5, [172], p. 88 (where the above argument may be found). So domains where $\bar{\partial}$ can be solved in the C^∞-category (at all form levels), and therefore pseudoconvex domains, are domains of holomorphy. Actually, the two classes of domains are the same, but the direction indicated here, namely that pseudoconvex domains are domains of holomorphy, is the hard part. (For a somewhat different approach to this implication, also based on $\bar{\partial}$-methods, see the remarks after the proof of Theorem 3.7.) This direction, historically known as the Levi problem, played a major role in the development of function theory of several complex variables. For more information and for historical comments, we refer the reader to [259], Chapter II, in particular Section 2.9 and the notes at the end of the chapter, and to [21], [274], [233], [211]. Lieb's account [211] also contains interesting biographical sketches, including photographs, of some of the early protagonists. For the original solutions, the references are Oka [246], [247], Bremermann [58], and Norguet [238] for bounded pseudoconvex domains in $\mathbb{C}^n$; and Grauert [149] for relatively compact strictly pseudoconvex domains in a complex manifold. [149], and implicitly already [247], use sheaf theoretic methods. For a development that uses neither sheaf theoretic nor $\bar{\partial}$ methods, see [134], Section V.6.

(iii) By Theorem 2.14, pseudoconvex domains have the property that $\bar{\partial}$ can be solved in $C^\infty_{(0,q+1)}(\Omega)$ for all form levels q, $0 \leq q \leq n-1$. Such domains are domains of holomorphy, by the previous remark. By the Cartan–Thullen convexity theory ([62], [259]), domains of holomorphy are pseudoconvex. Thus we have obtained more than the solution of the Levi problem, namely a cohomological characterization of pseudoconvex or holomorphy domains: a domain is pseudoconvex, equivalently, a domain of holomorphy, if and only if the q-th $\bar{\partial}$-cohomology vanishes for $1 \leq q \leq n$. If one 'only' wants to solve the Levi problem, one can get by with solvability of $\bar{\partial}$ on $(0,1)$-forms. It suffices in the above induction argument on the dimension to change the induction hypothesis to the effect that a pseudoconvex domain in $\mathbb{C}^n$ is a domain of holomorphy.

(iv) The cohomology in the previous remark, the Dolbeault cohomology, arises when the $\bar{\partial}$-complex is considered on forms with coefficients in $C^\infty(\Omega)$. In this chapter, the main focus has been on the complex on forms with coefficients in $\mathcal{L}^2(\Omega)$. The resulting cohomology, the $\mathcal{L}^2$-cohomology, does not necessarily agree with the Dolbeault cohomology. In particular, the cohomological characterization of pseudoconvexity from the previous remark remains true in this setting if the domain is the interior of its closure, but not in general. These matters are explained in [140].

Given Theorem 2.16 (and the context of this monograph), it is natural to ask whether there is a corresponding result with $\mathcal{L}^2$-bounds. The Ohsawa–Takegoshi extension theorem ([244], see also [240], [243]) gives such a result.

Theorem 2.17. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and H a complex hyperplane. There exists a constant C that depends only on the diameter of Ω (but not on H) such that for every holomorphic function f in $\mathcal{L}^2(H \cap \Omega)$ (with the induced Euclidean measure), there is a holomorphic function F on Ω with $F|_{H\cap\Omega} = f$ and*

$$\int_\Omega |F|^2 \leq C \int_{H\cap\Omega} |f|^2. \tag{2.117}$$

Moreover, the extension is given by a linear operator.

It is noteworthy that there are no transversality conditions on H. Like Theorem 2.16, Theorem 2.17 also allows for considerable generalizations. For example, H can be allowed to be a pure dimensional closed complex submanifold of $\mathbb{C}^n$, see [240, 242]. For extensions of certain sections of suitable vector bundles, see [216], [224], [300], and their references. For limits on what is possible, we refer the reader to [114], [115]. For example, the submanifold H cannot be allowed to have singularities, nor need there be extension with sup-norm bounds.

Proof of Theorem 2.17. The proof is based on Proposition 2.15, as amended in Remark (ii) following its proof; the argument follows [223], [243] (compare also [275], [35]).

We first note that the 'moreover' part is for free. Indeed, if there always is an extension with the required estimate, then it can be made linear by taking the unique extension of minimal norm, i.e., by projecting onto the orthogonal complement of the subspace of $\mathcal{L}^2(\Omega) \cap \mathcal{O}(\Omega)$ consisting of those functions that vanish on H.

Ω is pseudoconvex, so can be exhausted by a sequence $\{\Omega_m\}_{m=1}^\infty$ of strictly pseudoconvex C^2 domains. It suffices to prove that for every $m \in \mathbb{N}$, and $f \in \mathcal{L}^2(H \cap \Omega_{m+2}) \cap \mathcal{O}(H \cap \Omega_{m+2})$, there is $F \in \mathcal{L}^2(\Omega_m) \cap \mathcal{O}(\Omega_m)$ which extends f and with $\|F\|_{\Omega_m}$ dominated, independently of m, by $\|f\|_{H\cap\Omega_{m+2}} \leq \|f\|_{H\cap\Omega}$. These extensions of f, continued by zero on $\Omega \setminus \Omega_m$, are then (uniformly) bounded in $\mathcal{L}^2(\Omega)$. The weak limit of a suitable subsequence is holomorphic, its $\mathcal{L}^2$-norm is dominated by $\|f\|_{H\cap\Omega}$, and because point evaluations are continuous in the weak topology on holomorphic functions, it extends f.

By Theorem 2.16, there is a holomorphic function $\widehat{F}$ on Ω with $\widehat{F}|_{H\cap\Omega} = f$. Now fix m. We modify $\widehat{F}$ on Ω_m. Since m is fixed, we will not use subscripts on various objects in the argument below that depend on m. The strategy for modifying $\widehat{F}$ is the same as in the proof of Theorem 2.16, but the details are considerably more involved due to the estimates that we require. We may again assume that $H = \{z_n = 0\}$. Let $\chi \in C^\infty([0,1])$, $\chi \geq 0$, $\chi = 1$ on $[0, 1/4]$ and $\chi = 0$ on $[3/4, 1]$, and for $\varepsilon > 0$, set $\chi_\varepsilon(z) = \chi(|z_n|/\varepsilon)$. We look for an extension F_ε on Ω_m in the form $F_\varepsilon = \chi_\varepsilon \widehat{F} + z_n h_\varepsilon$, where h_ε solves

$$\bar\partial h_\varepsilon = -\frac{\widehat{F}\,\bar\partial\chi_\varepsilon}{z_n} =: \alpha_\varepsilon. \tag{2.118}$$

Then F_ε will be holomorphic on Ω_m and its restriction to $H \cap \Omega_m$ will agree with f. Note that α_ε is smooth on $\overline{\Omega_m}$ ($\bar\partial\chi_\varepsilon$ vanishes when $|z_n| \leq \varepsilon/4$).

To see what kind of estimate we require on the solution to (2.118), first observe that $\|\alpha_\varepsilon\|^2_{\Omega_m} \lesssim \varepsilon^{-4}\int_{\Omega_m\cap\{|z_n|<\varepsilon\}} |\widehat{F}|^2$ (because $|\bar\partial\chi_\varepsilon| \approx \varepsilon^{-1}$ and on the support of $\bar\partial\chi_\varepsilon$, $|z_n| \approx \varepsilon$). Here, $\lesssim$ indicates an estimate with a constant that does not depend on ε. Now let ε be small enough so that $\Omega_m \cap \{|z_n| < \varepsilon\} \subseteq (H \cap \Omega_{m+1}) \times \{|z_n| < \varepsilon\} \subset\subset \Omega_{m+2}$. Then $\varepsilon^{-2}\int_{\Omega_m\cap\{|z_n|<\varepsilon\}} |\widehat{F}|^2 \leq \varepsilon^{-2}\int_{(H\cap\Omega_{m+1})\times\{|z_n|<\varepsilon\}} |\widehat{F}|^2 \to \|f\|^2_{H\cap\Omega_{m+1}}$ as $\varepsilon \to 0^+$. Therefore, we seek an estimate for h_ε of the form

$$\|z_n h_\varepsilon\|_{\Omega_m} \lesssim \varepsilon\|\alpha_\varepsilon\|_{\Omega_m}; \tag{2.119}$$

then, as $\varepsilon \to 0^+$,

$$\begin{aligned}\|F_\varepsilon\|^2_{\Omega_m} &\leq \|\chi_\varepsilon\widehat{F}\|^2_{\Omega_m} + \|z_n h_\varepsilon\|^2_{\Omega_m} \\ &\lesssim (\varepsilon^2+1)\left(\varepsilon^{-2}\int_{(H\cap\Omega_{m+1})\times\{|z_n|<\varepsilon\}} |\widehat{F}|^2\right) \\ &\to \|f\|^2_{H\cap\Omega_{m+1}} \leq \|f\|^2_{H\cap\Omega}.\end{aligned} \tag{2.120}$$

Thus a limit of a suitable subsequence of the F_ε will give an extension with the desired estimate. Estimate (2.119) will result from an advantageous choice of the auxiliary functions $a = a_\varepsilon$ and $b = b_\varepsilon$ in Proposition 2.15, with $\varphi = 0$. The factor z_n in (2.119) allows for the term $\sqrt{a_\varepsilon + e^{b_\varepsilon}}$ in the estimate for h_ε that results from (2.113) to tend to infinity (slow enough) as $z_n \to 0$.

We first choose $b_\varepsilon = 2\log a_\varepsilon$, so that $e^{b_\varepsilon} = a_\varepsilon^2$ in (2.113). In order to be able to absorb the term involving the gradient of a_ε in the expression for the matrix G (see (2.109)), we take a_ε in the form $a_\varepsilon = p_\varepsilon + \log p_\varepsilon$, with $p_\varepsilon \geq 1$. Then

$$\frac{\partial a_\varepsilon}{\partial\bar z_k} = \left(1 + \frac{1}{p_\varepsilon}\right)\frac{\partial p_\varepsilon}{\partial\bar z_k}, \tag{2.121}$$

and

$$\frac{\partial^2 a_\varepsilon}{\partial z_j\partial\bar z_k} = \left(1 + \frac{1}{p_\varepsilon}\right)\frac{\partial^2 p_\varepsilon}{\partial z_j\partial\bar z_k} - \frac{1}{p_\varepsilon^2}\frac{\partial p_\varepsilon}{\partial z_j}\frac{\partial p_\varepsilon}{\partial\bar z_k}. \tag{2.122}$$

So we need

$$(p_\varepsilon + \log p_\varepsilon)^{-2}\left(1 + \frac{1}{p_\varepsilon}\right)^2 \leq p_\varepsilon^{-2}. \tag{2.123}$$

(2.123) holds if $p_\varepsilon \geq e$. Now for the choice of p_ε. What we have left over from the negative Hessian of a_ε in the expression for G is $(1 + (1/p_\varepsilon))$ times the negative Hessian of p_ε. We choose

$$p_\varepsilon = M - \log(|z_n|^2 + \varepsilon^2), \tag{2.124}$$

where M is a constant big enough so that $p_\varepsilon \geq e$ on Ω_m. Then

$$-\left(1 + \frac{1}{p_\varepsilon}\right)\frac{\partial^2 p_\varepsilon}{\partial z_n\partial\bar z_n} = \left(1 + \frac{1}{p_\varepsilon}\right)\frac{\varepsilon^2}{(|z_n|^2+\varepsilon^2)^2} \geq \frac{\varepsilon^2}{(|z_n|^2+\varepsilon^2)^2}. \tag{2.125}$$

Note that when $|z_n| < \varepsilon$, the right-hand side of (2.125) is at least $(1/4)\varepsilon^{-2}$. With these choices, the only nonzero entry in G is G_{nn}. Also, the only nonzero component of α_ε is the $d\bar{z}_n$ component. Therefore, we may choose the subspace H_{α_ε} in the amended version of Proposition 2.15 (see Remark (ii) after its proof) as follows. We take H_{α_ε} to consist of those forms in $\ker(\bar{\partial}) \subseteq \mathcal{L}^2_{(0,1)}(\Omega_m)$ of the form $u = f d\bar{z}_n$ with f supported in $\{|z_n| \leq \varepsilon\}$. Then (2.125) and the lower estimate of $(1/4)\varepsilon^{-2}$ for its right-hand side show that the right-hand side in (2.113) is estimated from above by $4\varepsilon^2\|u\|^2$. Therefore, (2.113) gives a solution h_ε to (2.118) that satisfies the estimate

$$\left\| \frac{h_\varepsilon}{\sqrt{a_\varepsilon + a_\varepsilon^2}} \right\|_{\Omega_m} \leq 2\varepsilon \|\alpha_\varepsilon\|_{\Omega_m}. \tag{2.126}$$

Then $z_n h_\varepsilon$ satisfies

$$\begin{aligned} \|z_n h_\varepsilon\|_{\Omega_m} &= \left\| z_n \sqrt{a_\varepsilon + a_\varepsilon^2} \frac{h_\varepsilon}{\sqrt{a_\varepsilon + a_\varepsilon^2}} \right\|_{\Omega_m} \\ &\leq 2\varepsilon \left(\sup_{\Omega_m} \left\{ |z_n| \sqrt{a_\varepsilon + a_\varepsilon^2} \right\} \right) \|\alpha_\varepsilon\|_{\Omega_m}. \end{aligned} \tag{2.127}$$

It now only remains to note that $\sup_{\Omega_m} \left\{ |z_n| \sqrt{a_\varepsilon + a_\varepsilon^2} \right\}$ is bounded independently of m and ε by a constant that depends only on the diameter of Ω ($|z_n|$ controls the log-terms near $z_n = 0$). □

Remarks. (i) The same proof shows that one may take norms with weight $e^{-\varphi}$ in (2.117), where φ is plurisubharmonic on Ω. More generally, these ideas give results when different weights, subject to certain conditions, are used in the two $\mathcal{L}^2$-spaces.

(ii) We will use Theorem 2.17 in the proof of Theorem 4.21. For applications related to the Bergman kernel function, see for example [241], [136], [220], [112], [38]; applications to algebraic geometry may be found in [275], [98], [100], [243], [38].

3 Strictly pseudoconvex domains

In this chapter, we discuss only the basic aspects of the $\mathcal{L}^2$-Sobolev theory of the $\bar\partial$-Neumann problem on smooth strictly pseudoconvex domains. Much more is known: the condition of strict pseudoconvexity allows for the construction of integral kernels that solve $\bar\partial$ with strong mapping properties. In particular, analogues of the Sobolev estimates proved below hold for $\mathcal{L}^p$-Sobolev spaces and for Hölder spaces. We refer the reader to [151], [259], [15], [212], and to their references.

3.1 Estimates at the ground level

Consider the simplest case of Proposition 2.4, i.e., the case $a \equiv 1$ and $\varphi \equiv 1$. Then (2.24) immediately gives

$$\sum_K{}' \int_{b\Omega} \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar z_k} u_{jK}\overline{u_{kK}}\, d\sigma \le \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \tag{3.1}$$

for $u \in \mathrm{dom}(\bar\partial^*) \cap C^1_{(0,q)}(\bar\Omega)$. This turns out to be a very strong estimate if the quadratic form that appears inside the boundary integral is positive definite. This is the case when Ω is strictly pseudoconvex. If Ω is bounded, so that $b\Omega$ is compact, there then exists a constant $c > 0$, such that

$$\sum_{j,k} \frac{\partial^2\rho}{\partial z_j \partial \bar z_k}(z) w_j \overline{w_k} \ge c|w|^2, \quad z \in b\Omega,\ w \in T_z^{\mathbb{C}}(b\Omega). \tag{3.2}$$

Inserting this estimate into the left-hand side of (3.1) now shows that when Ω is strictly pseudoconvex, the boundary $\mathcal{L}^2$-norm of u is dominated by the $\mathcal{L}^2$-norms (on Ω) of $\bar\partial u$ and $\bar\partial^* u$. Modulo controlling the Laplacian (of the coefficients) of u, this boundary norm should control the Sobolev-$(1/2)$ norm of u. But in view of Lemma 2.11, the Sobolev-(-1) norm of Δu is also dominated by $\|\bar\partial u\| + \|\bar\partial^* u\|$. These observations immediately give the following proposition. Recall that we denote by $\|f\|_s$ the standard Sobolev-s norm (see e.g. [215]) on a bounded domain with C^∞ smooth boundary. For forms, $W^s_{(0,q)}(\Omega)$ is the Hilbert space of $(0,q)$-forms with coefficients in $W^s(\Omega)$, with inner product

$$\Big(\sum_J{}' u_J\, d\bar z_J, \sum_J{}' v_J\, d\bar z_J\Big) = \sum_J{}' (u_J, v_J)_s. \tag{3.3}$$

Proposition 3.1. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with boundary of class C^∞, $1 \le q \le n$. Then $\mathrm{dom}(\bar\partial_q) \cap \mathrm{dom}(\bar\partial^*_{q-1}) \subseteq W^{1/2}_{(0,q)}(\Omega)$, and there exists a constant C such that*

$$\|u\|_{1/2} \le C(\|\bar\partial u\| + \|\bar\partial^* u\|), \quad u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*), \tag{3.4}$$

$$\|N_q u\|_{1/2} \leq C\|u\|_{-1/2}, \tag{3.5}$$

and

$$\|\bar{\partial} N_q u\| + \|\bar{\partial}^* N_q u\| \leq C\|u\|_{-1/2}. \tag{3.6}$$

Note that (3.5) and (3.6) say that N_q, $\bar{\partial} N_q$, and $\bar{\partial}^* N_q$, originally only defined on $\mathcal{L}^2_{(0,q)}(\Omega)$, extend to bounded operators from $W^{-1/2}_{(0,q)}(\Omega)$ to $W^{1/2}_{(0,q)}(\Omega)$, $\mathcal{L}^2_{(0,q+1)}(\Omega)$, and $\mathcal{L}^2_{(0,q-1)}(\Omega)$, respectively.

Proof of Proposition 3.1. Assume first that $u \in \mathrm{dom}(\bar{\partial}^*) \cap C^1_{(0,q)}(\bar{\Omega})$. The Sobolev theory of the Laplacian (see for example [124], [215], [299], [294]) gives that

$$\|u\|^2_{1/2} \leq C(\|\Delta u\|^2_{-1} + \|u\|^2_{\mathcal{L}^2(b\Omega)}), \tag{3.7}$$

where Δu is the form obtained by letting Δ act componentwise. Estimating $\|u\|^2_{\mathcal{L}^2(b\Omega)}$ by (3.2) and (3.1), and $\|\Delta u\|^2_{-1}$ by

$$\|\Delta u\|^2_{-1} \lesssim \|\bar{\partial}\vartheta u\|^2_{-1} + \|\vartheta\bar{\partial} u\|^2_{-1} \lesssim \|\bar{\partial}^* u\|^2 + \|\bar{\partial} u\|^2$$

($u \in \mathrm{dom}(\bar{\partial}^*)$, so $\vartheta u = \bar{\partial}^* u$), (3.4) follows.

If u is only assumed in $\mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$, then it can be approximated in the graph norm by forms smooth up to the boundary (Proposition 2.3). The case of (3.4) already proved then shows that the approximating forms form a Cauchy-sequence in $W^{1/2}_{(0,q)}(\Omega)$; passing to the limit shows that $u \in W^{1/2}_{(0,q)}(\Omega)$ and that (3.4) holds.

If $u \in \mathcal{L}^2_{(0,q)}(\Omega)$, then $N_q u \in W^{1/2}_{(0,q)}(\Omega)$ by (3.4) (since $N_q u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$), and moreover

$$\begin{aligned} \|N_q u\|^2_{1/2} &\lesssim \|\bar{\partial} N_q u\|^2 + \|\bar{\partial}^* N_q u\|^2 \\ &= Q(N_q u, N_q u) \\ &= (u, N_q u) \lesssim \|N_q u\|_{1/2}\|u\|_{-1/2}. \end{aligned} \tag{3.8}$$

In the last inequality in (3.8) we have used the duality between $W^{1/2}_{(0,q)}(\Omega)$ and $W^{-1/2}_{(0,q)}(\Omega)$ (which results from the duality between $W^{1/2}(\Omega)$ and $W^{-1/2}(\Omega)$; for this latter duality, see [215]). (3.8) now gives firstly (3.5) (since we already know that $N_q u \in W^{1/2}_{(0,q)}(\Omega)$, i.e., $\|N_q u\|_{1/2}$ is finite), and then, secondly, (3.6). ☐

3.2 Estimates for operators related to the $\bar{\partial}$-Neumann operator

Before extending the estimates in Proposition 3.1 to higher Sobolev norms, we prove a useful lemma that says that operators related to N_q, such as $\bar{\partial} N_q$, $\partial^* N_q$, $\bar{\partial}\bar{\partial}^* N_q$, and $\bar{\partial}^*\bar{\partial} N_q = I - \bar{\partial}\bar{\partial}^* N_q$, are 'as good as N' with regard to Sobolev estimates. This does not hinge on strict pseudoconvexity, but we prove the lemma now because we will need some of the arguments in its proof in the proof of Theorem 3.4 below.

Lemma 3.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary. For each $k \in \mathbb{N}$, there exists a constant C such that whenever u and $N_q u \in C^\infty_{(0,q)}(\overline{\Omega})$, then*

$$\|\bar{\partial} N_q u\|_k + \|\bar{\partial}^* N_q u\|_k + \|\bar{\partial}\bar{\partial}^* N_q u\|_k + \|\bar{\partial}^*\bar{\partial} N_q u\|_k \le C(\|N_q u\|_k + \|u\|_k). \quad (3.9)$$

Proof. Note first that the case $k = 0$ follows from Theorem 2.9 and the fact that $u = \bar{\partial}\bar{\partial}^* N_q u + \bar{\partial}^*\bar{\partial} N_q u$ is an orthogonal decomposition. Note that at this level, the a priori assumption that $N_q u$ is smooth up to the boundary is not needed: all the relevant forms are known to be in $\mathcal{L}^2$.

When k is an integer that is at least 1, the idea is the following: via integration by parts, one makes the combination $\Box_q N_q u = u$ appear. First note that on a relatively compact open subset U of Ω, the estimates (3.9) follow from interior elliptic regularity: $\|N_q u\|_{W^{k+2}_{(0,q)}(U)} \lesssim \|u\|_k$. We argue by induction on k. The case $k = 0$ has been disposed of above. If D^{k+1} is a derivative of order $k+1$, then repeated application of Lemma 2.2 shows that there is a tangential differential operator T_{k+1} of order $k+1$, such that near $b\Omega$

$$D^{k+1}\bar{\partial} N_q u = T_{k+1}\bar{\partial} N_q u + L_k(\bar{\partial}^*\bar{\partial} N_q u) + M_k(\bar{\partial} N_q u), \quad (3.10)$$

where L_k and M_k are operators of order k. The last two terms on the right-hand side of (3.10) are controlled by the induction assumption. A similar formula holds for $D^{k+1}\bar{\partial}^* N_q u$. For $D^{k+1}\bar{\partial}^*\bar{\partial} N_q u$, we have

$$D^{k+1}\bar{\partial}^*\bar{\partial} N_q u = T_{k+1}\bar{\partial}^*\bar{\partial} N_q u + L_k(\bar{\partial}\bar{\partial}^*\bar{\partial} N_q u) + M_k(\bar{\partial}^*\bar{\partial} N_q u). \quad (3.11)$$

We have slightly abused notation: the operators in (3.10) and (3.11) are of the same type, but not necessarily the same. The last term in (3.11) is controlled by the induction assumption, the second to the last is equal to $L_k(\bar{\partial} u)$ (since $\bar{\partial}^*\bar{\partial} N_q u = u - \bar{\partial}\bar{\partial}^* N_q u$), hence is controlled by $\|u\|_{k+1}$. It follows that we only have to control $\|T_{k+1}\bar{\partial} N_q u\|$, $\|T_{k+1}\bar{\partial}^* N_q u\|$, and $\|T_{k+1}\bar{\partial}^*\bar{\partial} N_q u\|$, where T_{k+1} is a tangential differential operator of order $k+1$. Because of the transformation rule for coefficients under a change of frame, it does not matter with respect to which frame we let T act coefficientwise. Therefore, we may take T to be supported in a special boundary chart and to act componentwise in the special boundary frame (hence preserving $\mathrm{dom}(\bar{\partial}^*)$). Then

$$\begin{aligned}
&\|T_{k+1}\bar{\partial} N_q u\|^2 + \|T_{k+1}\bar{\partial}^* N_q u\|^2 \\
&\quad = (\bar{\partial} T_{k+1} N_q u, T_{k+1}\bar{\partial} N_q u) + (\bar{\partial}^* T_{k+1} N_q u, T_{k+1}\bar{\partial}^* N_q u) \\
&\qquad + O(\|N_q u\|_{k+1}(\|T_{k+1}\bar{\partial} N_q u\| + \|T_{k+1}\bar{\partial}^* N_q u\|)) \\
&\quad = (T_{k+1} N_q u, \bar{\partial}^* T_{k+1}\bar{\partial} N_q u) + (T_{k+1} N_q u, \bar{\partial} T_{k+1}\bar{\partial}^* N_q u) \\
&\qquad + O(\|N_q u\|_{k+1}(\|T_{k+1}\bar{\partial} N_q u\| + \|T_{k+1}\bar{\partial}^* N_q u\|)) \\
&\quad \lesssim |(T_{k+1} N_q u, T_{k+1} u)| + \|N_q u\|_k(\|\bar{\partial} N_q u\|_{k+1} + \|\bar{\partial}^* N_q u\|_{k+1}) \\
&\quad \lesssim \|N_q u\|_{k+1}(\|u\|_{k+1} + \|\bar{\partial} N_q u\|_{k+1} + \|\bar{\partial}^* N_q u\|_{k+1}).
\end{aligned} \quad (3.12)$$

Adding up the analogues of (3.12) over a set of boundary charts that cover $b\Omega$, and inserting the sum into the estimate for $\|\bar{\partial}N_q u\|_{k+1}^2 + \|\bar{\partial}^* N_q u\|_{k+1}^2$, we see that the terms on the right-hand side of (3.12) containing $\|\bar{\partial}N_q u\|_{k+1}$ and $\|\bar{\partial}^* N_q u\|_{k+1}$ can be absorbed, and (3.9) results. (We use the usual inequality $|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for $\|N_q u\|_{k+1}\|u\|_{k+1}$). This completes the induction for the norms of $\bar{\partial}N_q u$ and $\bar{\partial}^* N_q u$.

Similarly, we have

$$\begin{aligned}
&(T_{k+1}\bar{\partial}^*\bar{\partial}N_q u, T_{k+1}\bar{\partial}^*\bar{\partial}N_q u) \\
&\quad = (\bar{\partial}^* T_{k+1}\bar{\partial}N_q u, T_{k+1}\bar{\partial}\bar{\partial}N_q u) + O(\|\bar{\partial}N_q u\|_{k+1}\|\bar{\partial}^*\bar{\partial}N_q u\|_{k+1}) \\
&\quad = (T_{k+1}\bar{\partial}N_q u, \bar{\partial}T_{k+1}\bar{\partial}^*\bar{\partial}N_q u) + O(\|\bar{\partial}N_q u\|_{k+1}\|\bar{\partial}^*\bar{\partial}N_q u\|_{k+1}) \\
&\quad = (T_{k+1}\bar{\partial}N_q u, T_{k+1}\bar{\partial}\bar{\partial}^*\bar{\partial}N_q u) + O(\|\bar{\partial}N_q u\|_{k+1}\|\bar{\partial}^*\bar{\partial}N_q u\|_{k+1}) \\
&\quad = (T_{k+1}\bar{\partial}N_q u, T_{k+1}\bar{\partial}u) + O(\|\bar{\partial}N_q u\|_{k+1}\|\bar{\partial}^*\bar{\partial}N_q u\|_{k+1}).
\end{aligned} \tag{3.13}$$

Since we already know that $\|\bar{\partial}N_q u\|_{k+1} \lesssim \|N_q u\|_{k+1} + \|u\|_{k+1}$, the last term in (3.13) can be absorbed. In the second to the last term, we reverse the above process to move the $\bar{\partial}$ from the right to the left. The result is that this term equals

$$\begin{aligned}
&(T_{k+1}\bar{\partial}^*\bar{\partial}N_q u, T_{k+1}u) + O(\|\bar{\partial}N_q u\|_{k+1}\|u\|_{k+1}) \\
&\quad \lesssim \left(\|\bar{\partial}^*\bar{\partial}N_q u\|_{k+1} + \|\bar{\partial}N_q u\|_{k+1}\right)\|u\|_{k+1}.
\end{aligned} \tag{3.14}$$

Absorbing the term containing $\|\bar{\partial}^*\bar{\partial}N_q u\|_{k+1}$ and using again that $\|\bar{\partial}N_q u\|_{k+1}$ is already estimated completes the argument. The proof of Lemma 3.2 is complete. □

Corollary 3.3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary. Let $1 \leq q \leq n$. Assume that N_q maps $W^k_{(0,q)}(\Omega)$ continuously to itself, for all $k \in \mathbb{N}$. Then $\bar{\partial}N_q$, $\bar{\partial}^* N_q$, $\bar{\partial}\bar{\partial}^* N_q$ $(= P_q)$, and $\bar{\partial}^*\bar{\partial}N_q$ all map $W^k(\Omega)$ continuously to itself at the respective form levels, for all $k \in \mathbb{N}$.*

Note that saying that N_q maps $W^k_{(0,q)}(\Omega)$ to itself is the same as saying that it does so continuously. This follows immediately from the closed graph theorem. That the graph of N_q as an operator $W^k_{(0,q)}(\Omega) \to W^k_{(0,q)}(\Omega)$ *is* closed follows from the continuity of N_q in $\mathcal{L}^2_{(0,q)}(\Omega)$.

Proof of Corollary 3.3. $C^\infty_{(0,q)}(\overline{\Omega})$ is dense in $W^k_{(0,q)}(\Omega)$. For $u \in C^\infty_{(0,q)}(\overline{\Omega})$, $N_q u \in C^\infty_{(0,q)}(\overline{\Omega})$ by assumption (and by an application of the Sobolev lemma). Therefore, we have the estimates in (3.9), and by density, these estimates extend to $W^k_{(0,q)}(\Omega)$ (since the operators are continuous between the respective $\mathcal{L}^2$ spaces). □

Remark. Corollary 3.3 says in particular that Sobolev estimates for N_q imply corresponding Sobolev estimates for the Bergman projection P_q. A more precise result about the relationship between regularity for the $\bar{\partial}$-Neumann operators N_q and the

Bergman projection operators P_q holds: estimates for N_q are *equivalent* to estimates for the Bergman projection operators P_{q-1}, P_q, P_{q+1}, at the three levels $q-1$, q, and $q+1$. In particular, exact regularity of N_1 implies exact regularity of P_0; this implication is not covered by Corollary 3.3. This result of Boas and the author ([47]) will be discussed in Chapter 5 (see Theorem 5.5).

3.3 Sobolev estimates, elliptic regularization

We now return to the question of extending the estimates in Proposition 3.1 to higher Sobolev norms. (3.4) is a so-called *subelliptic* estimate: u is in a Sobolev space of higher order than $\bar{\partial}u$ and $\bar{\partial}^*u$, but the gain is less than the order of the operator $(\bar{\partial}\oplus\bar{\partial}^*)$ indicates (that is, less than 1). (3.4) entails a similar subelliptic gain at all levels s in the Sobolev scale ($s \geq 0$). Similarly, (3.5) indicates a subelliptic gain of 1 for the $\bar{\partial}$-Neumann operator, and this gain percolates up the Sobolev scale. More precisely, we have the following theorem, which is the main result of this chapter. It is due to Kohn ([188], [190], see also [230], [229]).

Theorem 3.4. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary. Let $1 \leq q \leq n$. For each nonnegative real number s, there is a constant C so that the following estimates hold for every $(0,q)$-form u:*

$$\|u\|_{s+1/2} \leq C(\|\bar{\partial}u\|_s + \|\bar{\partial}^*u\|_s), \quad u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*), \tag{3.15}$$

$$\|N_q u\|_{s+1} \leq C\|u\|_s, \tag{3.16}$$

$$\|\bar{\partial}N_q u\|_{s+1/2} + \|\bar{\partial}^* N_q u\|_{s+1/2} \leq C\|u\|_s. \tag{3.17}$$

We emphasize that these are genuine estimates; that is, the left-hand side is finite if the right-hand side is. In particular, the operators N_q, $\bar{\partial}^*N_q$, and $\bar{\partial}N_q$ are continuous between the respective Sobolev spaces.

We postpone the proof of Theorem 3.4 and first discuss the method of elliptic regularization, introduced in this context by Kohn and Nirenberg in [201]. It consists of modifying the quadratic form Q (see (2.58)) by a term which makes the form *coercive* (see below). As a result, the inverse $N_{\delta,q}$ of the selfadjoint operator $\Box_{\delta,q}$ defined by the modified form will gain two derivatives in Sobolev norms (see Proposition 3.5 below). In particular, it will map forms smooth up to the boundary to forms smooth up to the boundary. This property is very useful in proving Sobolev estimates, such as the ones in Theorem 3.4, where typically one has to absorb terms. For this, one needs to know that the terms to be absorbed are finite. For this same reason, elliptic regularization will also be used in Chapter 3.

For the time being, Ω is only assumed to be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $u \in W^1_{(0,q)}(\Omega) \cap \mathrm{dom}(\bar{\partial}^*)$: this is the form domain of the new form Q_δ defined by

$$Q_\delta(u,u) := \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \delta\|\nabla u\|^2, \tag{3.18}$$

where ∇u denotes the vector of all (first) derivatives of all components of u. The term coercive here refers to the fact that $Q_\delta(u,u)$ dominates the 1-norm of u. This fact has crucial consequences: the problem of inverting the operator $\Box_{\delta,q}$ associated to Q_δ becomes elliptic, and the inverse $N_{\delta,q}$ gains two derivatives in the Sobolev scale. In particular, if $u \in C^\infty_{(0,q)}(\overline{\Omega})$, then $N_{\delta,q}u \in C^\infty_{(0,q)}(\overline{\Omega})$.

The construction of $\Box_{\delta,q}$ and its inverse $N_{\delta,q}$ is analogous to that of $\Box_q$ and N_q given in Chapter 1. Denote by $\overline{X}$ the conjugate dual of $W^1_{(0,q)}(\Omega)\cap\mathrm{dom}(\bar\partial^*)$ (with norm induced by Q_δ, which is equivalent to $\|\cdot\|_1$ for δ fixed). Then we have the injections $W^1_{(0,q)}(\Omega)\cap\mathrm{dom}(\bar\partial^*) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega) \hookrightarrow \overline{X}$, and $\widetilde{\Box_{\delta,q}}: W^1_{(0,q)}(\Omega)\cap\mathrm{dom}(\bar\partial^*) \hookrightarrow \overline{X}$ is defined to be the canonical identification of $W^1_{(0,q)}(\Omega)\cap\mathrm{dom}(\bar\partial^*)$ with $\overline{X}$ (compare the discussion for $\Box_q$ in Chapter 1; note that the form domain of Q_δ, $W^1_{(0,q)}(\Omega)\cap\mathrm{dom}(\bar\partial^*)$, is dense in $\mathcal{L}^2_{(0,q)}(\Omega)$). Now we set $\mathrm{dom}(\Box_{\delta,q}) = \{u \in W^1_{(0,q)}(\Omega)\cap\mathrm{dom}(\bar\partial^*) \mid \widetilde{\Box_{\delta,q}}u \in \mathcal{L}^2_{(0,q)}(\Omega)\}$ and $\Box_{\delta,q}u = \widetilde{\Box_{\delta,q}}u, u \in \mathrm{dom}(\Box_{\delta,q})$. By the general theory of selfadjoint operators associated to quadratic forms (see for example [260], Theorem VIII.15), $\Box_{\delta,q}$ is the unique selfadjoint operator on $\mathcal{L}^2_{(0,q)}(\Omega)$ with

$$(\Box_{\delta,q}u, v) = Q_\delta(u,v), \quad u \in \mathrm{dom}(\Box_{\delta,q}),\ v \in W^1_{(0,q)}(\Omega)\cap\mathrm{dom}(\bar\partial^*). \tag{3.19}$$

To compute $\Box_{\delta,q}u$ for $u \in \mathrm{dom}(\Box_{\delta,q})$, take v to be a smooth compactly supported q-form in Ω. Then

$$\begin{aligned} Q_\delta(u,v) &= (\bar\partial u, \bar\partial v) + (\bar\partial^* u, \bar\partial^* v) + \delta(\nabla u, \nabla v) \\ &= (\vartheta\bar\partial u, v) + (\bar\partial\bar\partial^* u, v) - \delta(\Delta u, v), \end{aligned} \tag{3.20}$$

by integration by parts (with the pairings taken in the sense of distributions when necessary, Δu is computed coefficientwise, see Lemma 2.11). This gives

$$(\Box_{\delta,q}u, v) = Q_\delta(u,v) = -\left(\left(\frac{1}{4}+\delta\right)\Delta u, v\right) \tag{3.21}$$

for v smooth and compactly supported. Such v are dense in $\mathcal{L}^2_{(0,q)}(\Omega)$, so it follows that

$$\Box_{\delta,q}u = -\left(\frac{1}{4}+\delta\right)\Delta u, \quad u \in \mathrm{dom}(\Box_{\delta,q}). \tag{3.22}$$

(3.22) not withstanding, $\Box_{\delta,q}$ is not a multiple of $\Box_q$: the domains are different, because the free boundary condition has changed. Indeed, repeating the computation in (3.20) with $v \in C^\infty_{(0,q)}(\overline{\Omega})\cap\mathrm{dom}(\bar\partial^*)$ and taking the boundary terms into account shows that $u \in C^2_{(0,q)}(\overline{\Omega})$ belongs to $\mathrm{dom}(\Box_{\delta,q})$ if and only if $u \in \mathrm{dom}(\bar\partial^*)$ and

$$(\bar\partial u)_{\mathrm{norm}} + \delta(\partial u/\partial\nu)_{\mathrm{Tan}} = 0 \quad \text{on } b\Omega. \tag{3.23}$$

Here, $(\partial/\partial\nu)$ acts componentwise in the Euclidean frame, $(\bar\partial u)_{\mathrm{norm}}$ denotes the normal component of $\bar\partial u$ (a $(0,q)$-form), and $(\partial u/\partial\nu)_{\mathrm{Tan}}$ denotes the tangential part of $(\partial u/\partial\nu)$

(see the discussion before the statement of Lemma 2.12; here, a normalized defining function is to be taken in the definition of u_{norm}). Note that for $\delta = 0$, (3.23) reverts back to just the free boundary condition in the $\bar{\partial}$-Neumann problem. A less ad hoc discussion of the free boundary condition (3.23) may be found in [295], Section 12.5.

We now define $N_{\delta,q}$. Note that although we are not using the analog of the embedding j_q explicitly, the definition used here is equivalent to that given for N_q in Chapter 1. For $u \in \mathcal{L}^2_{(0,q)}(\Omega)$, the pairing $v \to (u, v)$ defines a continuous conjugate linear functional on $W^1_{(0,q)}(\Omega) \cap \operatorname{dom}(\bar{\partial}^*)$, since

$$|(u, v)| \leq \|u\| \, \|v\| \leq \left(\frac{D^2 e}{q}\right)^{1/2} \|u\| (Q_\delta(v, v))^{1/2} \tag{3.24}$$

by Proposition 2.7. Consequently there is a unique element $N_{\delta,q} u \in W^1_{(0,q)}(\Omega) \cap \operatorname{dom}(\bar{\partial}^*)$ such that

$$(u, v) = Q_\delta(N_{\delta,q} u, v). \tag{3.25}$$

Then $N_{\delta,q}$ maps $\mathcal{L}^2_{(0,q)}(\Omega)$ onto $\operatorname{dom}(\Box_{\delta,q})$, and

$$\Box_{\delta,q} N_{\delta,q} u = u, \quad u \in \mathcal{L}^2_{(0,q)}(\Omega), \tag{3.26}$$

and

$$N_{\delta,q} \Box_{\delta,q} u = u, \quad u \in \operatorname{dom}(\Box_{\delta,q}), \tag{3.27}$$

that is, $N_{\delta,q}$ inverts $\Box_{\delta,q}$. That $N_{\delta,q}$ is selfadjoint can be seen in the same way that selfadjointness for N_q was proved. Note that by (3.24), the norm of $N_{\delta,q}$ as an operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to $W^1_{(0,q)}(\Omega) \cap \operatorname{dom}(\bar{\partial}^*)$, provided with the norm $\|v\| = Q_\delta(v, v)^{1/2}$, is bounded independently of δ by $(D^2 e/q)^{1/2}$. In particular, the norm of $N_{\delta,q}$ on $\mathcal{L}^2_{(0,q)}(\Omega)$ is bounded by $D^2 e/q$, independently of $\delta \geq 0$ (again in view of Proposition 2.7).

We now indicate the role played by the coercivity of Q_δ, that is, by the fact that Q_δ majorizes the 1-norm. If $v \in W^1_{(0,q)}(\Omega) \cap \operatorname{dom}(\bar{\partial}^*)$, then

$$\|v\|_1 \lesssim (Q_\delta(v, v))^{1/2} \approx \|\widetilde{\Box_{\delta,q}} v\|_{\overline{X}}. \tag{3.28}$$

Because the norm in $W^1_{(0,q)}(\Omega) \cap \operatorname{dom}(\bar{\partial}^*)$ is the 1-norm, the norm in $\overline{X}$ is roughly speaking like a (-1)-norm, so that (3.28) represents an (elliptic) gain of two derivatives for $\widetilde{\Box_{\delta,q}}$. The constants in (3.28) depend of course on δ and blow up as $\delta \to 0^+$. By contrast, $Q = Q_0$ does not majorize the 1-norm (or any Sobolev norm with positive index, if Ω is merely assumed pseudoconvex), and no estimate like (3.28) holds for $\widetilde{\Box_q}$. Once one has (3.28), it is a matter of lifting the estimate to higher Sobolev norms to get a gain of two derivatives for $N_{\delta,q}$, and this can be achieved by classical methods.

Proposition 3.5. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $\delta > 0$, $1 \leq q \leq n$, $s \geq 0$. Then $N_{\delta,q}$ maps $W^s_{(0,q)}(\Omega)$ continuously to $W^{s+2}_{(0,q)}(\Omega)$.*

Proof. The boundary conditions $u \in \mathrm{dom}(\bar{\partial}^*)$ together with the free boundary condition (3.23) are regular boundary conditions (when $\delta > 0$) for $\Box_{\delta,q}$ in the sense of Section 5.11 in [294], and the general results there consequently imply the proposition. However, the situation is more elementary, and (as pointed out in [295], Section 12.5) one can obtain a proof by following the arguments in the proof of the corresponding result for the Neumann problem for the Laplacian (the d-Neumann problem), as presented in [294], Section 5.7.

The starting point is (3.28). We first prove the Sobolev estimate in Proposition 3.5 at the ground level $s = 0$:

$$\|N_{\delta,q}u\|_2 \leq C_{\delta,q}\|u\|. \tag{3.29}$$

Actually, we will prove (3.29) in the form

$$\|u\|_2 \leq C_{\delta,q}\|\Box_{\delta,q}u\|, \quad u \in \mathrm{dom}(\Box_{\delta,q}). \tag{3.30}$$

The first observation is that if $\chi \in C^\infty(\overline{\Omega})$ and its normal derivative $\partial\chi/\partial\nu$ satisfies a suitable condition on $b\Omega$, then multiplication by χ preserves the domain of $\Box_{\delta,q}$. Let $u \in \mathrm{dom}(\Box_{\delta,q})$, $v \in \mathrm{dom}(\bar{\partial}^*) \cap W^1_{(0,q)}(\Omega)$. Then

$$\begin{aligned} &|\langle\widetilde{\Box_{\delta,q}}(\chi u), v\rangle| \\ &\quad= \left|(\bar{\partial}(\chi u), \bar{\partial}v) + (\bar{\partial}^*(\chi u), \bar{\partial}^*v) + \delta(\nabla(\chi u), \nabla v)\right| \\ &\quad\leq \left|(\chi\bar{\partial}u, \bar{\partial}v) + (\chi\bar{\partial}^*u, \bar{\partial}^*v) + \delta(\chi\nabla u, \nabla v)\right| \\ &\qquad+ \left|(\bar{\partial}\chi \wedge u, \bar{\partial}v) + \Big(\sum_K(\partial\chi/\partial z_j)u_{j,K}d\bar{z}_K, \bar{\partial}^*v\Big) + \delta((\nabla\chi)u, \nabla v)\right|. \end{aligned} \tag{3.31}$$

In the middle term on the last line of (3.31), $\bar{\partial}^*$ can be integrated by parts as $\bar{\partial}$ to the left side. As a result, this term is dominated by $\|u\|_1\|v\|$. In the other two terms, integration by parts also yields two terms that are dominated by $\|u\|_1\|v\|$; however, there are now also two boundary terms. Combining these two terms into one boundary integral, we have

$$\int_{b\Omega}\left((\bar{\partial}\chi \wedge u)_N + \delta\frac{\partial\chi}{\partial\nu}u, v\right)d\sigma, \tag{3.32}$$

where $(\cdot)_N$ denotes the normal component of a form (compare (3.20) and (3.23)). Note that both u and v are in $W^1_{(0,q)}(\Omega)$, so their coefficients have traces in $W^{1/2}(b\Omega)$, and the integrations by parts are justified (see e.g. [215], [124], [294], [299]). Because $u_N = 0$ on $b\Omega$, (3.32) equals

$$\int_{b\Omega}\left((\overline{L_n}\chi + \delta\frac{\partial\chi}{\partial\nu})u_T, v\right)d\sigma, \tag{3.33}$$

where for the moment L_n is the complex normal normalized so that $|\operatorname{Im} L_n| = 1$ on $b\Omega$. Then $\overline{L_n}\chi = (1/2)\,((\partial\chi/\partial\nu) + i\operatorname{Im} L_n\chi)$. Therefore, if

$$((1/2) + \delta)(\partial\chi/\partial\nu) + (1/2)i\operatorname{Im} L_n\chi = 0 \quad \text{on } b\Omega, \tag{3.34}$$

the (boundary) integral in (3.33) (hence in (3.32)) equals zero. This means that the last line in (3.31) is dominated by $\|u\|_1\|v\|$.

In the first line on the right-hand side of (3.31), we move the factor χ to the right side of the inner product in all three terms, and commute it with $\bar{\partial}$, $\bar{\partial}^*$, and ∇, respectively. The error caused by the commutators is bounded by $\|u\|_1\|v\|$. The resulting term equals $Q_\delta(u, \chi v) = \langle\widetilde{\Box_{\delta,q}}u, \chi v\rangle = (\Box_{\delta,q}u, \chi v)$ (note that $\chi v \in W^1_{(0,q)}(\Omega) \cap \mathrm{dom}(\bar{\partial}^*)$ since v is, and that $u \in \mathrm{dom}(\Box_{\delta,q})$).

Combining these estimates gives that

$$|\langle\widetilde{\Box_{\delta,q}}(\chi u), v\rangle| \lesssim (\|\Box_{\delta,q}u\| + \|u\|_1)\|v\|. \tag{3.35}$$

This says that if (3.34) is satisfied, then $\chi u \in \mathrm{dom}(\Box_{\delta,q})$, and

$$\|\Box_{\delta,q}(\chi u)\| \lesssim \|\Box_{\delta,q}u\| + \|u\|_1 \lesssim \|\Box_{\delta,q}u\| + \|\widetilde{\Box_{\delta,q}}u\| \lesssim \|\Box_{\delta,q}u\|. \tag{3.36}$$

The first inequality holds because of (3.35) and the fact that $W^1_{(0,q)}(\Omega) \cap \mathrm{dom}(\bar{\partial}^*)$ is dense in $\mathcal{L}^2_{(0,q)}(\Omega)$, the last inequality in (3.36) holds because $u \in \mathrm{dom}(\Box_{\delta,q})$. Note that once it is known that $\chi u \in \mathrm{dom}(\Box_{\delta,q})$, $\Box_{\delta,q}(\chi u)$ can be computed according to (3.22).

From this first observation, we conclude that in order to prove (3.30), it suffices to consider forms supported in a special boundary chart. To see this, start with a partition of unity $\{\chi_j\}$ of a neighborhood of $b\Omega$ subordinate to a cover by special boundary charts. For j fixed, choose a function $\psi_j \in C^\infty(\overline{\Omega})$ and compactly supported in the special boundary chart associated with χ_j, that equals zero on $b\Omega$ and whose normal derivative $\partial\psi_j/\partial\nu$ satisfies $((1/2)+\delta)(\partial\psi_j/\partial\nu) = -((1/2)+\delta)(\partial\chi_j/\partial\nu) - (1/2)i\,\mathrm{Im}\,L_n\chi_j$ on $b\Omega$. Then the functions $\chi_j + \psi_j$ are compactly supported in the respective special boundary chart, and they satisfy (3.34). Consequently, $(\chi_j + \psi_j)u \in \mathrm{dom}(\Box_{\delta,q})$, and (3.36) holds. Because $1/2 \le |\sum_j(\chi_j + \psi_j)| \le 2$ near $b\Omega$, we have for a suitably chosen $\varphi \in C_0^\infty(\Omega)$

$$\|u\|_2 \lesssim \|\varphi u\|_2 + \sum_j \|(\chi_j + \psi_j)u\|_2 \lesssim \|u\|_1 + \|\Box_{\delta,q}u\| \lesssim \|\Box_{\delta,q}u\|, \tag{3.37}$$

assuming that (3.30) holds for forms supported in a special boundary chart. We have also used the interior elliptic regularity for $\Box_{\delta,q}$ (in view of (3.22)), as in (2.83), and again the estimate $\|u\|_1 \approx \|\widetilde{\Box_{\delta,q}}u\|_{\overline{X}} \lesssim \|\Box_{\delta,q}u\|$ (in the last inequality in (3.37)).

So assume now that u is supported in a special boundary chart. We first estimate tangential derivatives. To this end, we employ the classical method of obtaining uniform estimates on difference quotients. These estimates will then imply that the corresponding derivative actually is in $\mathcal{L}^2$, or in the appropriate Sobolev space (see e.g. [122], Theorem 3, Section 5.8; [124], Theorem 6.21). Choose coordinates as in the proof of Proposition 2.3, that is, $(t_1, \dots, t_{2n-1})$ are coordinates on the boundary and r is the transverse coordinate. Sobolev norms may be computed and estimated in the local coordinates. Denote by D^h_j, $1 \le j \le 2n-1$, the difference quotient with respect to t_j, acting on forms coefficientwise in a special boundary frame associated to the special boundary chart. Then D_j^h preserves both $\mathrm{dom}(\bar\partial^*)$ and $W^1_{(0,q)}(\Omega)$. This is clear for $W^1_{(0,q)}(\Omega)$. For $\mathrm{dom}(\bar\partial^*)$, it can be seen as follows. The coefficients of forms in $W^1_{(0,q)}(\Omega)$ have traces on $b\Omega$ in $W^{1/2}(b\Omega)$. The arguments from Chapter 1 (see (2.9)) apply also in this more general context to show that $u \in W^1_{(0,q)}(\Omega)$ is in $\mathrm{dom}(\bar\partial^*)$ if and only if the normal component (defined via the traces of the coefficients) vanishes on the boundary. This last property is preserved under translations in the $(t_1, \dots, t_{2n-1})$ variables, hence under D_j^h. (Note that to see that the trace is zero, it does not matter whether we take the trace in Ω or in the local coordinates.) For $u \in \mathrm{dom}(\Box_{\delta,q})$, we want to estimate $\|D_j^h u\|_1$, uniformly in h. We do this by estimating the (equivalent) norm in $\tilde{X}$ of $\widetilde{\Box_{\delta,q}} D_j^h u$. We have

$$\begin{aligned}\left|\langle\widetilde{\Box_{\delta,q}} D_j^h u, v\rangle\right| &= \left|(\bar\partial D_j^h u, \bar\partial v) + (\bar\partial^* D_j^h u, \bar\partial^* v) + \delta|(\nabla D_j^h u, \nabla v)\right| \\ &\lesssim \|u\|_1\|v\|_1 + \left|(D_j^h\bar\partial u, \bar\partial v) + (D_j^h\bar\partial^* u, \bar\partial^* v) + \delta|(D_j^h\nabla u, \nabla v)\right|\end{aligned} \tag{3.38}$$

We have used that D_j^h applied to a function smooth up to the boundary is bounded uniformly in h (by the mean value theorem) in order to estimate the commutators between D_j^h and $\bar\partial$, $\bar\partial^*$, and ∇, respectively, applied to u. In the second line of (3.38), the D_j^h can be moved to the other side of the inner products as D_j^{-h} plus a 0-th order term. This can be seen by writing the inner products as integrals over $\mathbb{R}^n_-$ in the variables $(t_1, \dots, t_{2n-1}, r)$, where $\mathbb{R}^n_-$ denotes the half-space where $r < 0$. Upon also commuting D_j^{-h} with $\bar\partial$, $\bar\partial^*$, and ∇, respectively, we obtain

$$\begin{aligned}\left|\langle\widetilde{\Box_{\delta,q}} D_j^h u, v\rangle\right| &\lesssim \|u\|_1\|v\|_1 + |Q_\delta(u, D_j^{-h}v)| \\ &= \|u\|_1\|v\|_1 + |(\Box_{\delta,q}u, D_j^{-h}v)| \\ &\le \|u\|_1\|v\|_1 + \|\Box_{\delta,q}u\|\|D_j^{-h}v\|,\end{aligned} \tag{3.39}$$

with a constant that does not depend on h. We have used again that difference quotients of a function smooth up to the boundary are bounded uniformly in h. More generally, when $f \in W^1$, then the difference quotients D_j^h are bounded in $\mathcal{L}^2$ (uniformly in h), see for example [206], Lemma 7.5.6, [124], Theorem 6.47 (and its proof). In particular, the last term in (3.39) is bounded by $\|\Box_{\delta,q}u\|\|v\|_1$. Taking also into account that $\|u\|_1$ is dominated by $\|\widetilde{\Box_{\delta,q}}u\|_{\tilde{X}}$, hence by $\|\Box_{\delta,q}u\|$, we have that

$$\left|\langle\widetilde{\Box_{\delta,q}} D_j^h u, v\rangle\right| \lesssim \|\Box_{\delta,q}u\|\|v\|_1, \tag{3.40}$$

so that

$$\|D_j^h u\|_1 \lesssim \|\widetilde{\Box_{\delta,q}} D_j^h u\|_{\tilde{X}} \lesssim \|\Box_{\delta,q}u\|, \tag{3.41}$$

again with constants uniform in h. Since j is arbitrary, (3.41) implies that the tangential derivatives of u are all in $W^1(\Omega)$, and that $\|\nabla_T u\|_1 \lesssim \|\Box_{\delta,q}u\|$, where $\nabla_T u$ denotes the gradient with respect to $(t_1, \dots, t_{2n-1})$ (see again [124], Theorem 6.47).

To estimate the full Sobolev norm, observe that $\|u\|_2 \lesssim \|\nabla_T u\|_1 + \|\partial^2 u/\partial\nu^2\|$. Writing Δ in terms of tangential and normal derivatives in our local coordinates, we have

$$\Delta = \frac{\partial^2}{\partial\nu^2} + \frac{1}{2g}\frac{\partial g}{\partial\nu}\frac{\partial}{\partial\nu} + \frac{1}{\sqrt{g}}\sum_{j,k=1}^{2n-1}\frac{\partial}{\partial t_k}\left(g^{ik}\sqrt{g}\frac{\partial}{\partial t_j}\right), \tag{3.42}$$

where (g_{jk}) gives the Riemannian metric induced on the level sets of r by the (standard) Euclidean metric in $\mathbb{C}^n$, $(g^{jk}) = (g_{jk})^{-1}$, and $g = \det(g_{jk})$. Combining (3.42) with (3.22) shows that $\|\partial^2 u/\partial\nu^2\|$ is dominated by $\|\Box_{\delta,q}u\| + \|u\|_1 + \|\nabla_T u\|_1$, and thus, by what was shown above, by $\|\Box_{\delta,q}u\|$. Therefore, we have shown (3.30).

The next step in the proof of Proposition 3.5 consists of an induction to lift (3.30) to higher norms with integer index. We assume that for an integer $k \geq 0$, there is a constant $C_{\delta,k,q}$ such that if $u \in \mathrm{dom}(\Box_{\delta,q})$ and $\Box_{\delta,q}u \in W^k_{(0,q)}(\Omega)$, then $u \in W^{k+2}_{(0,q)}(\Omega)$, and

$$\|u\|_{k+2} \leq C_{\delta,k,q}\|\Box_{\delta,q}u\|_k. \tag{3.43}$$

(3.30) corresponds to the case $k = 0$. We must show that (3.43) holds with k replaced by $k+1$. The argument that it suffices to consider forms that are supported in a special boundary chart is essentially the same as above: in (the analogue of) (3.37) one uses the induction assumption (3.43) in lieu of $\|u\|_1 \lesssim \|\widetilde{\Box_{\delta,q}u}\|_{\tilde{X}}$. So assume that $u \in \mathrm{dom}(\Box_{\delta,q})$, $\Box_{\delta,q}u \in W^{k+1}_{(0,q)}(\Omega)$, and u is supported in a special boundary chart. We want to apply (3.43) to $D^h_j u$. For this, we must know that $D^h_j u \in \mathrm{dom}(\Box_{\delta,q})$. This is the case if the local coordinates are chosen carefully. Namely, we can choose $(t_1, \ldots, t_{2n-1})$ so that $\mathrm{Im}\, L_n$ corresponds to $\partial/\partial t_{2n-1}$ (since a single vector field can always be straightened). By the induction assumption, $u \in W^{k+2}_{(0,q)}(\Omega)$, with $k+2 \geq 2$. Therefore, the coefficients of u and their first order derivatives have traces on $b\Omega$ which are in $W^{k+1/2}(b\Omega)$; the same holds for $D^h_j u$. As above, $D^h_j u \in W^1_{(0,q)}(\Omega) \cap \mathrm{dom}(\bar\partial^*)$. The arguments that led to (3.23) work with boundary values taken in the sense of traces as well, and to see that $D^h_j u \in \mathrm{dom}(\Box_{\delta,q})$, it suffices to see that $D^h_j u$ satisfies (3.23) with boundary values taken as traces. Now $\partial/\partial\nu \approx \partial/\partial r$ commutes with D^h_j, and by our choice of coordinates, so does $\overline{L_n} \approx \partial/\partial r + a(t_1, \ldots, t_{2n-1}, r)\partial/\partial t_{2n-1}$ *at points of the boundary* (because $a(t_1, \ldots, t_{2n-1}, 0) \equiv -i$). Therefore

$$\begin{aligned}(\bar\partial D^h_j u)_{\mathrm{norm}} &+ \delta(\partial D^h_j u/\partial\nu)_{\mathrm{Tan}} \\ &= D^h_j\left((\bar\partial u)_{\mathrm{norm}} + \delta(\partial u/\partial\nu)_{\mathrm{Tan}}\right) \mod \nabla_T(u_{\mathrm{norm}}) \quad \text{on } b\Omega,\end{aligned} \tag{3.44}$$

where $\nabla_T(u_{\mathrm{norm}})$ denotes tangential derivatives of the normal component of u. Because $u \in \mathrm{dom}(\bar\partial^*)$, these latter derivatives are zero on the boundary. Because $u \in \mathrm{dom}(\Box_{\delta,q})$, $(\bar\partial u)_{\mathrm{norm}} + \delta(\partial u/\partial\nu)_{\mathrm{Tan}}$ is zero on the boundary, so that the first term on the right-hand side of (3.44) is zero as well. This shows that $D^h_j u \in \mathrm{dom}(\Box_{\delta,q})$. The rest of the argument is standard: the induction assumption (3.43) gives (with constants uniform in h):

$$\|D^h_j u\|_{k+2} \lesssim \|\Box_{\delta,q}D^h_j u\|_k \lesssim \|u\|_{k+2} + \|D^h_j \Box_{\delta,q}u\|_k \lesssim \|\Box_{\delta,q}u\|_{k+1}. \tag{3.45}$$

Consequently

$$\|\nabla_T u\|_{k+2} \le C_{\delta,k+1,q}\|\Box_{\delta,q}u\|_{k+1}. \tag{3.46}$$

It remains to estimate $\|\partial^{k+3}/\partial\nu^{k+3}u\| \lesssim \|\partial^2/\partial\nu^2 u\|_{k+1}$. This is done again using (3.42). This completes the induction and the proof of Proposition 3.5 when s is a non-negative integer.

The final step in the proof consists of noting that the result for general $s \ge 0$ follows from that for integer s by interpolation of linear operators. The Sobolev spaces form a so called interpolation scale; as a result, an operator that is continuous from $W^{s_j}(\Omega)$ to $W^{t_j}(\Omega)$, $j = 1, 2$, is also continuous from $W^{\vartheta s_1+(1-\vartheta)s_2}(\Omega)$ to $W^{\vartheta t_1+(1-\vartheta)t_2}(\Omega)$, $0 \le \vartheta \le 1$ (say for $s_j, t_j \ge 0$, $j = 1, 2$). For this, see for example [215], Chapter 1, [294], Chapter 4; the reader interested in a thorough treatment of interpolation in its own right may consult [33]. This result carries over to the Sobolev spaces of forms; in particular, continuity of $N_{\delta,q}$ from $W^k_{(0,q)}(\Omega)$ to $W^{k+2}_{(0,q)}(\Omega)$ for all integers $k \ge 0$ implies continuity from $W^{k+\vartheta}_{(0,q)}(\Omega)$ to $W^{k+\vartheta+2}_{(0,q)}(\Omega)$, $0 \le \vartheta \le 1$. This completes the proof of Proposition 3.5. □

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. Let $u \in C^\infty_{(0,q)}(\overline{\Omega})$. Then by Proposition 3.5 and the Sobolev imbedding theorem, $N_{\delta,q}u \in C^\infty_{(0,q)}(\overline{\Omega})$. We will first show the estimates (under this assumption)

$$\|N_{\delta,q}u\|_{k+1/2} \le C_{k,q}\|u\|_{k-1/2}, \quad k = 1, 2, \ldots \tag{3.47}$$

with $C_{k,q}$ independent of δ. The restriction $k \ge 1$ takes care of some technicalities that would otherwise arise. We have, again by the Sobolev theory of the Laplacian,

$$\begin{aligned}\|N_{\delta,q}u\|^2_{k+1/2} &\lesssim \|\Delta N_{\delta,q}u\|^2_{k-3/2} + \|N_{\delta,q}u\|^2_{k,b\Omega}\\ &\lesssim \|u\|^2_{k-3/2} + \|N_{\delta,q}u\|^2_{k,b\Omega}.\end{aligned} \tag{3.48}$$

To estimate $\|N_{\delta,q}u\|_{k,b\Omega}$, we consider vector fields that are tangential at the boundary and let them act coefficientwise in special boundary frames. More precisely, returning to coordinates in a special boundary chart like the ones used in the proof of Proposition 3.5, we let T_j denote one of the $\partial/\partial t_j$, $1 \le j \le 2n-1$, but multiplied by a suitable cutoff function so that it is defined on $\overline{\Omega}$. It then suffices to estimate $\|T^\alpha N_{\delta,q}u\|_{0,b\Omega}$, where $\alpha = (\alpha_1, \ldots, \alpha_{2n-1})$ is a multi-index of length k, $T^\alpha = T_1^{\alpha_1}\ldots T_{2n-1}^{\alpha_{2n-1}}$, and the T_j are as described. Note that $T^\alpha N_{\delta,q}$ is then also in $\mathrm{dom}(\bar\partial^*)$. In the following estimates, all constants are independent of δ, unless otherwise indicated. (3.1) and (3.2) give

$$\begin{aligned}\|T^\alpha N_{\delta,q}u\|^2_{0,b\Omega} &\lesssim \|\bar\partial T^\alpha N_{\delta,q}u\|^2 + \|\bar\partial^* T^\alpha N_{\delta,q}u\|^2\\ &\le Q_\delta(T^\alpha N_{\delta,q}u, T^\alpha N_{\delta,q}u).\end{aligned} \tag{3.49}$$

The right-hand side of (3.49) is dominated as follows:

$$Q_\delta(T^\alpha N_{\delta,q}u, T^\alpha N_{\delta,q}u) \lesssim |(T^\alpha u, T^\alpha N_{\delta,q}u)| + \|N_{\delta,q}u\|^2_k; \tag{3.50}$$

this is a special case of Lemma 3.1 in [201] (or see Lemma 2.4.2 in [125]). We postpone the proof of this special case and continue with the proof of Theorem 3.4.

We would like to estimate $|(T^\alpha u, T^\alpha N_{\delta,q}u)|$ by $\|u\|_{k-1/2}\|N_{\delta,q}u\|_{k+1/2}$. This requires some extra care because derivatives of order k do not in general map $W^{k-1/2}(\Omega)$ to $W^{-1/2}(\Omega)$ (see [215], Section 12.8, Chapter 1, for a discussion of this). However, tangential derivatives are not a problem. We indicate how to obtain the desired estimate. Duality between $W^{-1/2}(\mathbb{R}^{2n-1})$ and $W^{1/2}(\mathbb{R}^{2n-1})$ gives, for suitable r_0:

$$
\begin{aligned}
&|(T^\alpha u, T^\alpha N_{\delta,q}u)| \\
&\quad\lesssim \int_{r_0}^{0} \|u(\cdot,r)\|_{W^{k-1/2}(\mathbb{R}^{2n-1})}\|N_{\delta,q}u(\cdot,r)\|_{W^{k+1/2}(\mathbb{R}^{2n-1})}\,dr \qquad (3.51)\\
&\quad\le \left(\int_{r_0}^{0} \|u(\cdot,r)\|^2_{W^{k-1/2}(\mathbb{R}^{2n-1})}dr\right)^{1/2}\left(\int_{r_0}^{0} \|N_{\delta,q}u(\cdot,r)\|^2_{W^{k+1/2}(\mathbb{R}^{2n-1})}dr\right)^{1/2},
\end{aligned}
$$

where the norms have their obvious meaning. The two terms in the second line of (3.51) are the norms of u and of $N_{\delta,q}u$, viewed as elements of $\mathcal{L}^2([r_0,0])$ with values in $W^{k-1/2}(\mathbb{R}^{2n-1})$ and $\mathcal{L}^2([r_0,0])$ with values in $W^{k+1/2}(\mathbb{R}^{2n-1})$, respectively. It now suffices to observe that these norms are dominated by $\|u\|_{k-1/2}$ and $\|N_{\delta,q}u\|_{k+1/2}$, respectively. (For example, use a continuous extension operator to $\mathbb{R}^{2n}$; for compactly supported functions in $\mathbb{R}^{2n}$, the corresponding statement is obvious from the description of the Sobolev spaces via the Fourier transform.) Thus $|(T^\alpha u, T^\alpha N_{\delta,q}u)| \lesssim \|u\|_{k-1/2}\|N_{\delta,q}u\|_{k+1/2}$.

Combining this estimate with interpolation between $\|N_{\delta,q}u\|_{k+1/2}$ and $\|N_{\delta,q}u\|_0$ (i.e., $\|N_{\delta,q}u\|_k^2 \le \text{s.c.}\|N_{\delta,q}u\|^2_{k+1/2} + \text{l.c.}\|N_{\delta,q}u\|_0^2$, see [215], Theorem 16.3, [124], Chapter 6, Section A, [148], Theorem 7.28) gives that the right-hand side of (3.50) is dominated by

$$
\begin{aligned}
\|u\|_{k-1/2}\|N_{\delta,q}u\|_{k+1/2} + \text{s.c.}\|N_{\delta,q}u\|^2_{k+1/2} + \text{l.c.}\|u\|_0^2 \\
\lesssim \text{s.c.}\|N_{\delta,q}u\|^2_{k+1/2} + \text{l.c.}\|u\|^2_{k-1/2},
\end{aligned}
\qquad (3.52)
$$

where s.c.and l.c. denote a small and a large constant, respectively. The boundary of Ω can be covered by finitely many boundary charts, and we have estimates (3.49)–(3.52) for each of these charts. Adding up these estimates, combining with (3.48), and choosing s.c. small enough lets us absorb $\|N_{\delta,q}u\|^2_{k+1/2}$ into the left-hand side of the estimate. The estimate we obtain is (3.47).

This was for $u \in C^\infty_{(0,q)}(\overline{\Omega})$ (hence $N_{\delta,q}u \in C^\infty_{(0,q)}(\overline{\Omega})$), but since $C^\infty_{(0,q)}(\overline{\Omega})$ is dense in $W^{k-1/2}_{(0,q)}(\overline{\Omega})$, and $N_{\delta,q}$ is continuous in $\mathcal{L}^2_{(0,q)}(\Omega)$, it follows that (3.47) for k fixed, holds for all $u \in W^{k-1/2}_{(0,q)}(\Omega)$.

We now claim that N_q inherits these estimates 'upon letting $\delta \to 0^+$'. The procedure is the following. Take $u \in W^{k-1/2}_{(0,q)}(\Omega)$. Then $\{N_{\delta,q}u \mid 0 < \delta < \delta_0\}$ is a bounded set in $W^{k+1/2}_{(0,q)}(\Omega)$, because the estimate (3.47) is independent of δ. Consequently, there is a subsequence $\{N_{\delta_n,q}u\}$ that converges weakly in $W^{k+1/2}_{(0,q)}(\Omega)$ to a limit, say $\hat{u}$

as $n \to \infty$. Of course $\|\hat{u}\|_{k+1/2} \le C_{k,q}\|u\|_{k-1/2}$. But $\hat{u} = N_q u$. This can be seen as follows. If $N_{\delta_n,q}u \to \hat{u}$ weakly in $W^{k+1/2}_{(0,q)}(\Omega)$, then first derivatives of $N_{\delta_n,q}u$ converge weakly to the corresponding first derivatives of $\hat{u}$ in $\mathcal{L}^2(\Omega)$ (as $k \ge 1$). From this observation it follows easily that $\hat{u} \in \mathrm{dom}(\bar{\partial}^*)$. Now take $v \in W^1_{(0,q)}(\Omega) \cap \mathrm{dom}(\bar{\partial}^*)$. Then

$$Q(N_q u, v) = (u, v) = Q_\delta(N_{\delta,q}u, v), \tag{3.53}$$

by definition of N_q and $N_{\delta,q}$, respectively. Furthermore, the last term in (3.53) tends to $Q(\hat{u}, v)$ as $n \to \infty$, by the observation about the weak convergence of derivatives of $N_{\delta_n,q}u$. Therefore, (3.53) gives, upon letting n tend to infinity, that $Q(N_q u, v) = Q(\hat{u}, v)$. This in turn implies $N_q u = \hat{u}$, because $W^1_{(0,q)}(\Omega) \cap \mathrm{dom}(\bar{\partial}^*)$ is dense in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$, by Proposition 2.3. This completes the proof of the Sobolev estimates (3.16) for N_q for $s = k - 1/2$, $k = 1, 2, \dots$, and the case $k = 0$ is covered by Proposition 3.1.

The estimates for $\|\bar{\partial}N_q\|_k$ and $\|\bar{\partial}^* N_q\|_k$ follow from those for $\|N_q\|_k$ and the proof of Lemma 3.2. Take $u \in C^\infty_{(0,q)}(\overline{\Omega})$. Then, by what we have already shown, $N_q u \in C^\infty_{(0,q)}(\overline{\Omega})$. In the proof of Lemma 3.2 for estimating $\|\bar{\partial}N_q u\|_k + \|\bar{\partial}^* N_q u\|_k$, $\|u\|_k$ enters only in estimating the term $(T^\alpha N_q u, T^\alpha u)$, which there is estimated by $\|N_q u\|_k \|u\|_k$. If instead we use (3.51) (and the discussion following it), we can estimate the term by $\|N_q u\|_{k+1/2}\|u\|_{k-1/2} \lesssim \|u\|^2_{k-1/2}$ (by the estimate for $N_q u$). A density argument as above extends these estimates to all of $W^{k-1/2}_{(0,q)}(\Omega)$.

The estimates at all levels s in (3.16) and (3.17) follow by interpolation from those shown (although the index $-1/2$ requires caution, $W^{-1/2}(\Omega)$ interpolates 'correctly' with $W^{k-1/2}(\Omega)$, see for example Theorem 12.5 in [215]).

It remains to prove the estimates in (3.15). They are consequences of what is already proved: for $u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$, write $u = \bar{\partial}\bar{\partial}^* N_q u + \bar{\partial}^*\bar{\partial}N_q u = \bar{\partial}N_{q-1}\bar{\partial}^* u + \bar{\partial}^* N_{q+1}\bar{\partial}u$ and apply (3.17). Actually, there is one case that is not formally covered by this; when $q = 1$, the $\bar{\partial}$-Neumann operator N_0 arises, which we have not discussed (but see [81]). To avoid using N_0, we can write $\bar{\partial}\bar{\partial}^* N_1 u = (N_1\bar{\partial})\bar{\partial}^* u$; recall from Chapter 1 that $N_1\bar{\partial}$, originally defined only on $\mathrm{dom}(\bar{\partial})$, extends as a continuous operator to all of $\mathcal{L}^2(\Omega)$. Now one checks along the lines of the ideas above that this operator satisfies $\|N_1\bar{\partial}v\|_{k+1/2} \lesssim \|v\|_k$. This completes the proof of Theorem 3.4, subject to the proof of (3.50), which we will address after the following remark.

Remark. In proving Theorem 3.4, one can work directly with Sobolev spaces of general real index. (3.48) remains valid, and one can then use tangential Bessel potential operators in (3.49) for the norm on the boundary; see the appendix in [125] for the relevant properties of these operators. The arguments above carry over essentially verbatim.

The proof of (3.50) will involve commutators between an operator A and powers of an operator T. The following purely algebraic formula is very useful for dealing with such commutators (see e.g. [103], Lemma 2, p. 418):

$$
\begin{aligned}
[A, T^k] &= \sum_{j=1}^{k} \binom{k}{j} \underbrace{[\dots[[A,T],T]\dots,T]}_{j\text{-fold}} T^{k-j} \\
&= \sum_{j=1}^{k} \binom{k}{j} T^{k-j} \underbrace{[\dots[A,T],T]\dots,T]}_{j\text{-fold}}.
\end{aligned}
\tag{3.54}
$$

To prove (3.54), write $AT^k = T^k A + \sum_{j=0}^{k-1} T^j [A,T] T^{k-j-1}$, obtained by commuting T^k past A, one factor T at a time. Successively commuting all factors of T to one side of the commutator in each term of the sum produces iterated commutators, and keeping track of how many j-fold iterated ones there are gives (3.54).

To establish (3.50), we use a now familiar strategy: we commute T^α with $\bar\partial$, $\bar\partial^*$, and ∇, and integrate by parts in order to make a term of the form $Q_\delta(N_{\delta,q}u, v) = (u, v)$ appear. Thus

$$
\begin{aligned}
Q_\delta(T^\alpha N_{\delta,q}u, T^\alpha N_{\delta,q}u) &= (\bar\partial T^\alpha N_{\delta,q}u, \bar\partial T^\alpha N_{\delta,q}u) + (\bar\partial^* T^\alpha N_{\delta,q}u, \bar\partial^* T^\alpha N_{\delta,q}u) \\
&\quad + \delta(\nabla T^\alpha N_{\delta,q}u, \nabla T^\alpha N_{\delta,q}u) \\
&= (T^\alpha \bar\partial N_{\delta,q}u, \bar\partial T^\alpha N_{\delta,q}u) + (T^\alpha \bar\partial^* N_{\delta,q}u, \bar\partial^* T^\alpha N_{\delta,q}u) \\
&\quad + \delta(T^\alpha \nabla N_{\delta,q}u, \nabla T^\alpha N_{\delta,q}u) \\
&\quad + O(\|N_{\delta,q}u\|_k^2) + \frac{1}{2} Q_\delta(T^\alpha N_{\delta,q}u, T^\alpha N_{\delta,q}u).
\end{aligned}
\tag{3.55}
$$

The last term in (3.55) can be absorbed into the left-hand side. The $O(\cdot)$-term has constants independent of δ. The sum of the first three terms on the right-hand side of (3.55) equals

$$
\begin{aligned}
&(\bar\partial N_{\delta,q}u, (T^\alpha)^* \bar\partial T^\alpha N_{\delta,q}u) + (\bar\partial^* N_{\delta,q}u, (T^\alpha)^* \bar\partial^* T^\alpha N_{\delta,q}u) \\
&\quad + \delta(\nabla N_{\delta,q}u, (T^\alpha)^* \nabla T^\alpha N_{\delta,q}u),
\end{aligned}
\tag{3.56}
$$

where $(T^\alpha)^*$ denotes the $\mathcal{L}^2$-adjoint of T^α. This is again a tangential operator of order k. Now

$$
\begin{aligned}
&(\bar\partial N_{\delta,q}u, (T^\alpha)^* \bar\partial T^\alpha N_{\delta,q}u) \\
&\quad = (\bar\partial N_{\delta,q}u, [(T^\alpha)^*, \bar\partial] T^\alpha N_{\delta,q}u) + (\bar\partial N_{\delta,q}u, \bar\partial (T^\alpha)^* T^\alpha N_{\delta,q}u) \\
&\quad = (\bar\partial N_{\delta,q}u, \big[[(T^\alpha)^*, \bar\partial], T^\alpha\big] N_{\delta,q}u) + (\bar\partial N_{\delta,q}u, T^\alpha [(T^\alpha)^*, \bar\partial] N_{\delta,q}u) \\
&\qquad + (\bar\partial N_{\delta,q}u, \bar\partial (T^\alpha)^* T^\alpha N_{\delta,q}u)
\end{aligned}
\tag{3.57}
$$

We analyze the commutator in the first term on the right-hand side of (3.57). (3.54) with $A = [(T^\alpha)^*, \bar\partial]$ gives that $[(T^\alpha)^*, \bar\partial] T^\alpha = (T_1^{\alpha_1} [(T^\alpha)^*, \bar\partial] +$ terms of the form

$T_1^{\alpha_1-j_1}B_{1,j_1})T_2^{\alpha_2}\dots T_{2n-1}^{\alpha_{2n-1}}$, where B_{1,j_1} is an operator of order k and $1 \le j_1 \le \alpha_1$. We can continue in this fashion with $T_2^{\alpha_2}$, etc., to obtain that $\big[[(T^\alpha)^*, \bar\partial], T^\alpha\big]$ is a sum of terms of the form $T^\beta B_\beta$, where B_β is an operator of order k and $|\beta| \le k-1$. Integrating T^β by parts to the left side of the inner products shows that the first term on the right-hand side of (3.57) is dominated by $\|\bar\partial N_{\delta,q}u\|_{k-1}\|N_{\delta,q}u\|_k \lesssim \|N_{\delta,q}u\|_k^2$.

In the second term on the right-hand side of (3.57), we integrate T^α by parts to the left and commute it with $\bar\partial$. Modulo error terms that are $O(\|N_{\delta,q}u\|_k^2)$, this term therefore equals, in absolute value,

$$\begin{aligned}
&|(\bar\partial T^\alpha N_{\delta,q}u, [(T^\alpha)^*, \bar\partial]N_{\delta,q}u)| \\
&\quad\lesssim \text{s.c.}\|\bar\partial T^\alpha N_{\delta,q}u\|^2 + \text{l.c.}\|N_{\delta,q}u\|_k^2 \\
&\quad\le \text{s.c.}Q_\delta(T^\alpha N_{\delta,q}u, T^\alpha N_{\delta,q}u) + \text{l.c.}\|N_{\delta,q}u\|_k^2.
\end{aligned} \tag{3.58}$$

The first term on the right can be absorbed into the left-hand side of (3.55).

Similar arguments apply to the other two terms in (3.56). This leaves us with the sum of the third term on the right-hand side of (3.57) and its analogues for $\bar\partial^*$ and ∇ (coming from the second and third term in (3.56)). This sum equals $Q_\delta(N_{\delta,q}u, (T^\alpha)^*T^\alpha N_{\delta,q}u) = (u, (T^\alpha)^*T^\alpha N_{\delta,q}u) = (T^\alpha u, T^\alpha N_{\delta,q}u)$. This completes the proof of (3.50) and of Theorem 3.4. ☐

Theorem 3.4 is sharp in the following sense: if (3.15) holds for (0, 1)-forms, then the domain is necessarily strictly pseudoconvex. Namely, we then have $\|u\|_{\mathcal{L}^2(b\Omega)} \lesssim \|u\|_{1/2} + \|\Delta u\|_{-1} \lesssim \|\bar\partial u\| + \|\bar\partial^* u\|$ for a $(0,1)$-form $u \in \mathrm{dom}(\bar\partial^*) \cap C^\infty_{(0,1)}(\bar\Omega)$. Here, we have used that although the trace is not generally defined on $W_{1/2}(\Omega)$, it is on functions whose Laplacian is in $W^{-1}(\Omega)$, and it satisfies the estimate in the first inequality. (This is a standard consequence of the elliptic theory of the Laplacian: write $u = u_1 + u_2$, with u_1 harmonic and $u_2 \in W_0^1(\Omega)$, cf. [215]). Therefore an estimate called 'the basic estimate' in [125], page 22, holds. By Theorem 3.2.2 in [125], this implies that Ω is strictly pseudoconvex. (Note that here Ω is assumed pseudoconvex; therefore the possibility that the Levi form has $q+1 = 2$ negative eigenvalues, also allowed by Theorem 3.2.2, cannot occur.)

Remark. One can say more if one considers nonisotropic Sobolev spaces (i.e., different 'numbers of derivatives' in different directions). For example, if Ω is a smooth bounded strictly pseudoconvex domain, $u \in \mathrm{dom}(\bar\partial)\cap\mathrm{dom}(\bar\partial^*) \subseteq \mathcal{L}^2_{(0,q)}(\Omega)$, and $X = \sum_{j=1}^n a_j(\partial/\partial z_j)$ is a smooth vector field on $\bar\Omega$ such that $\sum_{j=1}^n a_j(p)(\partial\rho/\partial z_j)(p) = 0$ for each boundary point p, then

$$\|Xu\|^2 \le C_X(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2); \tag{3.59}$$

that is, $\|\bar\partial u\| + \|\bar\partial^* u\|$ controls a full derivative of type $(1,0)$ in complex tangential directions. Since all derivatives of type $(0,1)$ are dominated by $\|\bar\partial u\| + \|\bar\partial^* u\|$ (in view of (2.46) and the remark following the proof of Lemma 2.6), we obtain that all complex tangential derivatives are dominated by $\|\bar{\partial}u\| + \|\bar{\partial}^* u\|$ (but only half a derivative is when all directions are allowed). The derivation of (3.59) starts with integration by parts as in (5.46) in the proof of Lemma 5.6. The computation there shows that

$$\|Xu\|^2 = \|\bar{X}u\|^2 - \int_\Omega ([\bar{X}, X]u)\bar{u} + O\left((\|Xu\| + \|\bar{X}u\|)\|u\|\right). \tag{3.60}$$

The term $\|Xu\|\|u\| \leq \text{s.c.}\|Xu\|^2 + \text{l.c.}\|u\|^2$ can be absorbed (note that in view of Proposition 2.3, it suffices to prove (3.59) for forms smooth up to the boundary, so that the norms involved are finite). By Lemma 2.6 and Proposition 2.7, $\|\bar{X}u\|^2 + \|\bar{X}u\|\|u\|$ is dominated by the right-hand side of (3.59). The remaining term $\int_\Omega ([\bar{X}, X]u)\bar{u}$ is dominated by $\|u\|^2_{1/2}$; this is analogous to the discussion concerning the first term on the right-hand side of (3.50) above, because $[\bar{X}, X]$ is a *tangential* first order operator. By Proposition 3.1, $\|u\|^2_{1/2}$ is dominated by the right-hand side of (3.59), and we are done.

One can now modify the methods in the proof of Theorem 3.4 to obtain that the $\bar{\partial}$-Neumann operator gains two complex tangential derivatives (on a strictly pseudoconvex domain). Just as in the isotropic case, the nonisotropic estimates hold for $\mathcal{L}^p$-Sobolev and Hölder spaces as well, but their proofs require machinery that is considerably more sophisticated ([151], [15], Remark 2 on page 187).

3.4 Pseudolocal estimates

The estimates in Theorem 3.4 are strong enough that they localize, modulo a weak global term. Such estimates are called ‘pseudolocal’ estimates; they go back to [188], [190] (compare also [201]).

Theorem 3.6. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary. Let $1 \leq q \leq n$, and fix two smooth cutoff functions φ_1, φ_2, such that φ_2 is identically equal to one in a neighborhood of the support of φ_1. For every $s \geq 0$, there exists a constant C so that the following estimates hold for every $(0,q)$-form u:*

$$\|\varphi_1 u\|_{s+1/2} \leq C(\|\varphi_2\bar{\partial}u\|_s + \|\varphi_2\bar{\partial}^* u\|_s + \|\bar{\partial}u\| + \|\bar{\partial}^* u\|),\quad u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*); \tag{3.61}$$

$$\|\varphi_1 N_q u\|_{s+1} \leq C\|\varphi_2 u\|_s + \|u\|; \tag{3.62}$$

$$\|\varphi_1\bar{\partial}N_q u\|_{s+1/2} + \|\varphi_1\bar{\partial}^* N_q u\|_{s+1/2} \leq C\|\varphi_2 u\|_s + \|u\|. \tag{3.63}$$

Remark. One can also obtain pseudolocal estimates for P_{q-1} from those for $\bar{\partial}^* N_q$; details are in Remark (iv) after the proof of Theorem 5.5.

Proof of Theorem 3.6. We first show (3.61). Note that $\varphi_1 u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. The estimates in the proof of (3.61) do not involve absorbing terms; they are genuine in the sense that the left-hand side is finite (i.e., the forms *are* in the Sobolev space in question) when the right-hand side is. Theorem 3.4 gives

$$\begin{aligned}\|\varphi_1 u\|_{s+1/2} &\lesssim \|\bar\partial(\varphi_1 u)\|_s + \|\bar\partial^*(\varphi_2 u)\|_s \\ &\lesssim \|\varphi_1\bar\partial u\|_s + \|\varphi_1\bar\partial^* u\|_s + \|\chi_1 u\|_s,\end{aligned} \tag{3.64}$$

where χ_1 is a cutoff function equal to one in a neighborhood of the support of φ_1 and with support contained in the set where φ_2 is equal to one. The last term on the right-hand side of (3.64) is of the same type as the term on the left-hand side, but with the Sobolev index lowered by $1/2$. We can therefore argue inductively to obtain

$$\begin{aligned}\|\varphi_1 u\|_{s+1/2} &\lesssim \|\varphi_1\bar\partial u\|_s + \|\varphi_1\bar\partial^* u\|_s + \|\chi_1\bar\partial u\|_{s-1/2} + \|\chi_1\bar\partial^* u\|_{s-1/2} + \cdots \\ &\quad \cdots + \|\chi_k\bar\partial u\|_{s-k/2} + \|\chi_k\bar\partial^* u\|_{s-k/2} + \|\chi_{k+1} u\|_{s-k/2} \\ &\lesssim \|\varphi_2\bar\partial u\|_s + \|\varphi_2\bar\partial^* u\|_s + \|\bar\partial u\|_0 + \|\bar\partial^* u\|_0,\end{aligned} \tag{3.65}$$

where k is the first integer such that $0 < s-k/2 \leq 1/2$ and the χ_j, $2 \leq j \leq k+1$, are nested cutoff functions chosen so that χ_j equals one in a neighborhood of the support of χ_{j-1} and such that φ_2 equals one on the support of χ_{k+1}. The last inequality in (3.65) follows from the support properties of the cutoff functions ($\|\chi_j\bar\partial u\|_{s-j/2} = \|\chi_j\varphi_2\bar\partial u\|_{s-j/2} \lesssim \|\varphi_2\bar\partial u\|_{s-j/2} \lesssim \|\varphi_2\bar\partial u\|_s$; multiplication by a smooth function is continuous in Sobolev norms) and from Theorem 3.4 (or Proposition 3.1).

We now prove (3.62). We may assume without loss of generality that φ_1 and φ_2 are supported in a special boundary chart. In such a chart, we use coordinates as in the proof of Proposition 3.5 or in the proof of Theorem 3.4. The tangential Bessel potential operators Λ^s are defined via the Fourier transform in the tangential variables $(t_1, \ldots, t_{2n-1})$. Thus, for f supported in the special boundary chart, and in $\mathcal{L}^2(\Omega)$, set

$$\tilde f(\tau_1, \ldots, \tau_{2n-1}, r) = \int_{\mathbb{R}^{2n-1}} e^{i(\sum_{j=1}^{2n-1}\tau_j t_j)} f(t_1, \ldots, t_{2n-1}, r)\, dV(t_1, \ldots, t_{2n-1}). \tag{3.66}$$

f can be recovered from $\tilde f$ via the Fourier inversion formula in the tangential variables. Now define Λ^s via

$$\widetilde{\Lambda^s f}(\tau_1, \ldots, \tau_{2n-1}, r) = (1+|\tau|^2)^{s/2}\tilde f(\tau_1, \ldots, \tau_{2n-1}, r). \tag{3.67}$$

Note that while $\Lambda^s f$ is still supported in $\mathbb{R}^n_-$, its support in general is no longer compact (in $(\tau_1, \ldots, \tau_{2n-1})$). For properties of the operators Λ^s, we refer the reader to the Appendix of [125], where they are discussed in detail. The reason we need these operators here is that they control the Sobolev norms with respect to the tangential variables. For forms, Λ^s acts coefficientwise (in the special boundary chart we are working in). In particular, for u as above, $\|u\|_{s,b\Omega} \approx \|\Lambda^s u(\cdot,0)\|_{\mathcal{L}^2(\mathbb{R}^{2n-1})}$. Choose a cutoff function χ whose support is contained in the set where φ_2 equals one and which is equal to one on a neighborhood of the support of φ_1. We first prove (3.62) for

$u \in C^{\infty}_{(0,q)}(\overline{\Omega})$. Note that then also $N_q u \in C^{\infty}_{(0,q)}(\overline{\Omega})$, by Theorem 3.4. We have for $s \geq 0$,

$$\begin{aligned} \|\varphi_1 N_q u\|_{s+1} &\lesssim \|\Delta(\varphi_1 N_q u)\|_{s-1} + \|\varphi_1 N_q u\|_{s+1/2,b\Omega} \\ &\lesssim \|\varphi_1 u\|_{s-1} + \|\chi N_q u\|_s + \|\Lambda^{s+1/2}\varphi_1 N_q u(\cdot,0)\|_{\mathcal{L}^2(\mathbb{R}^{2n-1})}. \end{aligned} \tag{3.68}$$

We have used here that if σ is a derivative of φ_1, then $\sigma = \sigma\chi$, so that $\|\sigma N_q u\|_s \lesssim \|\chi N_q u\|_s$. As always, we also freely use the fact that Sobolev norms are equivalent under diffeomorphisms, so that it is immaterial whether norms are taken on Ω or on $\mathbb{R}^{2n}_{-}$. In order to proceed, we need to 'change' the last term in (3.68) to one that is supported in the boundary chart we are working in, so that the form corresponds to a form on Ω. To this end, observe that $\chi\Lambda^{s+1/2}$ maps forms supported in our special boundary chart to such forms, and that $\chi\Lambda^{s+1/2}\varphi_1 N_q u - \Lambda^{s+1/2}\varphi_1 N_q u = [\chi, \Lambda^{s+1/2}]\varphi_1 N_q u$ (because $\chi\varphi_1 = \varphi_1$). The commutator $[\chi, \Lambda^{s+1/2}]$ is a (pseudodifferential) operator of order $s - 1/2$ (see the Appendix in [125]). Consequently, (3.68) gives

$$\begin{aligned} \|\varphi_1 N_q u\|_{s+1} \lesssim \|\varphi_1 u\|_{s-1} &+ \|\chi N_q u\|_s \\ &+ \|\chi\Lambda^{s+1/2}\varphi_1 N_q u\|_{0,b\Omega} + \|\varphi_1 N_q u\|_{s-1/2,b\Omega}. \end{aligned} \tag{3.69}$$

Estimating the last term on the right-hand side of (3.69) by $\|\varphi_1 N_q u\|_s + \|\Delta\varphi_1 N_q u\|_{s-1}$ (using the same version of the trace theorem as in the discussion preceding the statement of Theorem 3.6; the $(s-1)$-norm, rather than the $(s-2)$-norm, is to cover the case where $s-2$ is less than -1) shows that it is dominated by the first two terms on the right-hand side of (3.69). With respect to the second to the last term on the right-hand side of (3.69), note that $\chi\Lambda^{s+1/2}\varphi_1 N_q u$ is in $C^{\infty}_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar\partial^*)$: $\chi\Lambda^{s+1/2}$ preserves smoothness up to the boundary (since $\Lambda^{s+1/2}$ preserves $C^{\infty}(\mathbb{R}^{2n-1})$, and it suffices to check that the normal component vanishes on the boundary to confirm membership in $\mathrm{dom}(\bar\partial^*)$. Thus we can use (3.1) and (3.2) to bound this term. In summary, we obtain

$$\begin{aligned} \|\varphi_1 N_q u\|_{s+1} \lesssim \|\varphi_1 u\|_{s-1} &+ \|\chi N_q u\|_s \\ &+ Q(\chi\Lambda^{s+1/2}\varphi_1 N_q u, \chi\Lambda^{s+1/2}\varphi_1 N_q u). \end{aligned} \tag{3.70}$$

An estimate analogous to (3.50) holds for the last term in (3.70), at the expense of possibly 'enlarging' (the support of) the cutoff function χ. This is verified in the same way as (3.50), but paying attention to supports and using commutator properties of the tangential Bessel potential operators analogous to the corresponding properties for differential operators ([125], Appendix) (or see Lemma 3.1 in [201], Lemma 2.4.2 in [125]). So choose χ_1 identically equal to one in a neighborhood of the support of χ and supported in the set where φ_2 equals one. Then, as in the proof of Theorem 3.4,

$$\begin{aligned} &Q(\chi\Lambda^{s+1/2}\varphi_1 N_q u, \chi\Lambda^{s+1/2}\varphi_1 N_q u) \\ &\qquad\lesssim |(\chi\Lambda^{s+1/2}\varphi_1 u, \chi\Lambda^{s+1/2}\varphi_1 N_q u)| + \|\chi_1 N_q u\|^2_{s+1/2} \\ &\qquad\lesssim \|\varphi_1 u\|_s \|\varphi_1 N_q u\|_{s+1} + \|\chi_1 N_q u\|^2_{s+1/2} \\ &\qquad\lesssim \text{s.c.}\|\varphi_1 N_q u\|^2_{s+1} + \text{l.c.}\|\varphi_1 u\|^2_s + \|\chi_1 N_q u\|^2_{s+1/2}. \end{aligned} \tag{3.71}$$

Combining (3.70) and (3.71) gives, after absorbing the term s.c.$\|\varphi_1 N_q u\|^2_{s+1}$ into the left-hand side,

$$\|\varphi_1 N_q u\|_{s+1} \lesssim \|\varphi_1 u\|_s + \|\chi_1 N_q u\|_{s+1/2}. \tag{3.72}$$

As in (3.64), the second term on the right-hand side of (3.72) is of the same kind as the term of the left-hand side, but with the Sobolev index lowered by $1/2$. An induction similar to the one used to derive (3.65) from (3.64) gives

$$\begin{aligned} \|\varphi_1 N_q u\|_{s+1} &\lesssim \|\varphi_2 u\|_s + \|\chi_{k+1} N_q u\|_{s-k/2-1/2} \\ &\lesssim \|\varphi_2 u\|_s + \|N_q u\|_1 \lesssim \|\varphi_2 u\|_s + \|u\|_0, \end{aligned} \tag{3.73}$$

where k is the first integer such that $0 < s - k/2 - 1/2 \leq 1$ and χ_{k+1} is a suitable cutoff function supported on the set where φ_2 equals one. The last inequality in (3.73) comes from Theorem 3.4 ((3.16) for $s = 0$). This concludes the proof of (3.62) when u is smooth on the closure of Ω.

For general u, we use an approximation argument in combination with splitting u into a part near the support of φ_1 and a part away from that support. Choose a cutoff function χ so that φ_1, χ, and φ_2 are nested. Also choose a sequence of forms $u_n \in C^\infty_{(0,q)}(\overline{\Omega})$ such that $u_n \to u$ in $\mathcal{L}^2_{(0,q)}(\Omega)$. Then (since the supports of χ and φ_1 are disjoint) by the case of (3.62) already proved, $\|\varphi_1 N_q (1-\chi)(u_n - u_m)\|_{s+1} \lesssim \|(1-\chi)(u_n - u_m)\|_0$ (with the constant independent of n, m). Consequently, the sequence $\{\varphi_1 N_q (1-\chi) u_n\}_{n=1}^\infty$ is Cauchy in $W^{s+1}_{(0,q)}(\Omega)$. Since it converges in $\mathcal{L}^2_{(0,q)}(\Omega)$ to $\varphi_1 N_q (1-\chi)u$, $\varphi_1 N_q (1-\chi)u \in W^{s+1}_{(0,q)}(\Omega)$, and $\|\varphi_1 N_q (1-\chi)u\|_{s+1} \lesssim \|(1-\chi)u\|_0$. Combining this with the global estimate from Theorem 3.4, applied to $\|\varphi_1 N_q (\chi u)\|_{s+1}$, gives the desired estimate:

$$\begin{aligned} \|\varphi_1 N_q u\|_{s+1} &\leq \|\varphi_1 N_q (\chi u)\|_{s+1} + \|\varphi_1 N_q (1-\chi)u\|_{s+1} \\ &\lesssim \|\chi u\|_s + \|(1-\chi)u\|_0 \lesssim \|\varphi_2 u\|_s + \|u\|_0 \end{aligned} \tag{3.74}$$

(since $\chi u = \chi \varphi_2 u$).

It remains to prove (3.63). This can be done along the lines of the proof of the corresponding statement in Theorem 3.4, i.e., by modifying the proof of Lemma 3.2 in a similar way, by arguments that are by now familiar. Consequently, the details are left to the reader. This completes the proof of Theorem 3.6. □

Remarks. (i) The reader has probably noticed that it is not necessary to formulate and prove Theorems 3.4 and 3.6 separately: one can prove (the stronger) Theorem 3.6 directly (still using elliptic regularization). However, breaking up the proof into the global part and then the localization may more clearly show what the issues are at each step.

(ii) Scrutinizing the proofs of Theorems 3.4 and 3.6 shows that in Theorem 3.6, it suffices to assume that the smooth domain Ω is strictly pseudoconvex on (and hence in a neighborhood of) the support of φ_1.

(iii) (3.62) was derived from (3.72) by a rather obvious induction. It is interesting that there is a more sophisticated iteration scheme that allows to have the global term with a norm of negative index; this is not obvious, because (3.72) is available only for $s \geq 0$. For this, see Appendix A in [41].

The estimates in Theorems 3.4 and 3.6 are subelliptic estimates: there is a gain in Sobolev norms, but it is less than the order of the operator. More generally, the $\bar{\partial}$-Neumann problem is said to satisfy a subelliptic estimate of order ε for $(0,q)$-forms at a boundary point P, if there is a neighborhood V of P such that for all $(0,q)$-forms u supported in $V \cap \overline{\Omega}$ and smooth there, the following estimate holds:

$$\|u\|_{\varepsilon} \leq C(\|\bar{\partial}u\| + \|\bar{\partial}^{*}u\|), \quad u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^{*}), \tag{3.75}$$

where C does not depend on u. Of course, $\varepsilon > 0$, and from what was said above, $\varepsilon \leq 1/2$. Work of D'Angelo ([87], [89]) and Catlin ([66], [67], [70]) provides a characterization of when a subelliptic estimate holds, in terms of the geometric notion of finite type. For $q = 1$, the appropriate notion of finite type of a boundary point P is that there should be a (finite) bound for the order of contact with $b\Omega$ at P of one-dimensional analytic varieties. Then a subelliptic estimate holds at P for $(0,1)$-forms if and only if P is of finite type. A subelliptic estimate implies the pseudolocal estimates from Theorem 3.6, but with gains of ε, 2ε, and ε, respectively. As in the $\varepsilon = 1/2$ case, these estimates go back to [201]; they can be proved along the lines of the argument above. In addition to the references given so far, the reader may consult [69], [92], [94], [95], [168], [278], [236], [156], [71], [73], [180] and their references for further information on subellipticity and finite type.

3.5 Singular holomorphic functions

We conclude this chapter with another application of $\bar{\partial}$-techniques to several complex variables: existence of holomorphic functions with singular behavior at a specified boundary point P.

Theorem 3.7. *Let Ω be a smooth bounded strictly pseudoconvex domain, $P \in b\Omega$. Then there exists a holomorphic function f on Ω, smooth up to the boundary except at P, with $\lim_{\Omega \ni z \to P} |f(z)| = \infty$.*

Proof. We follow [188], Section 9. Denote by ρ a defining function for Ω which is strictly plurisubharmonic in a neighborhood of $b\Omega$. Such a defining function can be found in the form $\rho(z) = e^{a\sigma(z)} - 1$, where σ is an arbitrary defining function, by choosing the real constant a big enough; see Proposition 2.14 in [259] for details. Set

$$g(z) := \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p)(p_j - z_j) - \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p)(p_j - z_j)(p_k - z_k) \tag{3.76}$$

g is a holomorphic polynomial of degree 2 (called the Levi polynomial of ρ at p), and $g(p) = 0$. The Taylor expansion of ρ at p together with the fact that ρ is strictly plurisubharmonic implies that the zero set of g stays outside $\overline{\Omega}$ near P (except at P): there is a ball $B(P, r_0)$ s.th. $\{z|g(z) = 0\} \cap \overline{\Omega} \cap B(P, z_0) = \{P\}$ (see [259],Proposition 2.16 for details). Choose a cutoff function $\chi \in C_0^\infty(B(P, r_0))$ with $\chi \equiv 1$ on $B(P, r_0/2)$. Then

$$\alpha := \bar{\partial}\,(\chi(1/g)) \tag{3.77}$$

is in $C^\infty_{(0,1)}(\overline{\Omega})$. By Theorem 3.4, there is $h \in C^\infty(\overline{\Omega})$ with $\bar{\partial}h = \alpha$. Set

$$f(z) := h(z) - \chi(z)(1/g(z)). \tag{3.78}$$

f is holomorphic on Ω ($\bar{\partial} f = 0$), and f is smooth up to the boundary except at P (since both h and $\chi(1/g)$ are). Because h is also smooth up to P, we have that $\lim_{z \to P} |f(z)| = \infty$, and Theorem 3.7 is proved. ☐

Remarks. (i) Starting from Theorem 3.7, methods of classical function theory allow one to find holomorphic functions on Ω that cannot be continued past *any* boundary point: Ω is a domain of holomorphy. Furthermore, the classical methods also yield that the union of an increasing sequence of domains of holomorphy is a domain of holomorphy (this is the Behnke–Stein Theorem, cf. [19], [20], and [301], Section 16, Chapter III). Consequently, Theorem 3.7 (in tandem with classical function theory) provides another proof that a pseudoconvex domain is a domain of holomorphy, that is, another solution of the Levi problem (compare Remarks (ii) and (iii) after the proof of Theorem 2.16; there, we also give references to prior work and different approaches).

(ii) In the above proof, we have used the boundary regularity results on strictly pseudoconvex domains derived in Chapter 2. While this is convenient, it is not necessary. Observe that the form α in (3.77) is actually smooth in a neighborhood of $\overline{\Omega}$: the zero set of g stays away from $\overline{\Omega}$ on the support of $\bar{\partial}\chi$. So we can choose a smooth (even strictly) pseudoconvex domain Ω_1 that contains $\overline{\Omega}$ and such that $\alpha \in C^\infty_{(0,1)}(\overline{\Omega_1}) \subseteq \mathcal{L}^2_{(0,1)}(\Omega_1)$. By Corollary 2.10, there is $h \in \mathcal{L}^2(\Omega_1)$ with $\bar{\partial}h = \alpha$. By *interior* elliptic regularity of $\bar{\partial}$ (as in (2.85)), h is smooth on Ω_1, hence on $\overline{\Omega}$. Note that to conclude that $\bar{\partial}$ maps onto $\ker(\bar{\partial}) \subseteq \mathcal{L}^2_{(0,1)}(\Omega_1)$, all that is needed is (2.49) (essentially Lemma 2.6) for smoothly bounded domains. Also, the interior elliptic regularity used relies on little more than the elementary $\mathcal{L}^2$ theory of the Fourier transform. This proof comes from [257], [259], Section V.1.4, except that there Corollary 2.10 (or Lemma 2.6) is replaced by an elementary integral formula for an approximate solution operator to $\bar{\partial}$. This produces a proof of Theorem 3.7 (and hence a solution of the Levi problem) that minimizes the machinery used.

(iii) Another interesting variant of the argument in the proof of Theorem 3.7, taken from [192], shows that there are bounded (necessarily nonpseudoconvex) domains in $\mathbb{C}^n$, $n \geq 2$, where $\bar{\partial}\colon \mathcal{L}^2(\Omega) \to \mathcal{L}^2_{(0,1)}(\Omega)$ does not have closed range. Assume only that the domain is strictly pseudoconvex near the boundary point P, while the rest of the boundary is not assumed pseudoconvex. Instead, assume that $\bar{\partial}\colon \mathcal{L}^2(\Omega) \to \mathcal{L}^2_{(0,1)}(\Omega)$ has closed range. We still have the polynomial g. By taking a sufficiently high power of g, if necessary, we may assume that $1/g$ fails to be square integrable in $U \cap \Omega$ for any neighborhood U of P. Approximating g near P by translations of it in the direction of the normal at P, it is easy to see that the form α in (3.77) is in the closure of the range of $\bar{\partial}$, hence in the range of $\bar{\partial}$ (which is closed by assumption). Thus there is $h \in \mathcal{L}^2(\Omega)$ with $\bar{\partial} h = \alpha$. The holomorphic function $f = h - \chi(1/g)$ fails to be square integrable near P. Now choose Ω and P such that P belongs to a bounded component of $\mathbb{C}^n \setminus \Omega$ (for example, Ω can be a spherical shell with an inward bump on the inner boundary, and P a can be a strictly pseudoconvex point on the bump). In this case, Hartogs' theorem ([206], Theorem 1.2.6, [259], Theorem 2.1, Chapter IV) implies that the holomorphic function f extends holomorphically past P, and so is in particular square integrable near P, contradicting our earlier finding. Thus, for such Ω, the range of $\bar{\partial}\colon \mathcal{L}^2(\Omega) \to \mathcal{L}^2_{(0,1)}(\Omega)$ is not closed.

4 Compactness

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then we know from Chapter 2 that the $\bar{\partial}$-Neumann operator N_q is continuous form $\mathcal{L}^2_{(0,q)}(\Omega)$ to itself. In this chapter, we are interested in the situation when N_q is compact.

Whether or not the $\bar{\partial}$-Neumann operator is compact is relevant in a number of circumstances. Examples include the Fredholm theory of Toeplitz operators ([125], [167]), existence or non-existence of Henkin–Ramirez type kernels for solving $\bar{\partial}$ ([164], and certain C^* algebras of operators naturally associated to a domain ([266]). In the $\mathcal{L}^2$-Sobolev theory of the $\bar{\partial}$-Neumann problem, the relevance of compactness historically stems from the fact that it implies regularity in Sobolev spaces (i.e., global regularity, [201], Theorem 4.6 below). In addition, over the last ten years or so, it has become increasingly clear that global regularity is a very subtle property, and there are reasons to hope that compactness is perhaps more readily amenable to a characterization in terms of properties of the boundary. For example, compactness may be viewed as a limiting case of subellipticity (which gives compactness because $W^s_{(0,q)}(\Omega) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega)$ is compact, by Rellich's lemma, if Ω is a bounded domain) when the gain in the subelliptic estimate tends to zero, and subellipticity is indeed characterized by properties of the boundary, namely the property of finite type (for smooth domains; see the discussion in Section 3.4).

4.1 An application to Toeplitz operators

In the Fredholm theory of Toeplitz operators the crucial property is that commutators between the Bergman projection and certain multiplication operators are compact. This is an easy consequence of compactness of the $\bar{\partial}$-Neumann operator. Recall that P_q denotes the Bergman projection on $(0,q)$-forms, i.e., the orthogonal projection from $\mathcal{L}^2_{(0,q)}(\Omega)$ onto the subspace of $\bar{\partial}$ closed forms. The following proposition was essentially observed in [72].

Proposition 4.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that for some q, $0 \le q \le n-1$, the canonical solution operator $\bar{\partial}^* N_{q+1}$ is compact. Let f be continuous on $\overline{\Omega}$. Then the commutator $[P_q, f]$ is compact on $\mathcal{L}^2_{(0,q)}(\Omega)$.*

Note that compactness of $\bar{\partial}^* N_{q+1}$ follows from compactness of either N_q or N_{q+1}, in view of Proposition 4.2, equivalence of (i) and (iv), below.

Proof of Proposition 4.1. The proof we give is from [142]. Assume first that f is not only continuous on $\overline{\Omega}$, but that in addition f has first derivatives that are continuous and bounded on Ω. Let $u \in \mathcal{L}^2_{(0,q)}(\Omega)$ and set $u_1 := P_q u$, $u_2 := (\mathbb{1} - P_q)u$. Both P_q and $(\mathbb{1} - P_q)$ are continuous (they are orthogonal projections). Therefore, it suffices to show that $[P_q, f]$ is compact on both $\ker(\bar\partial)$ and $\ker(\bar\partial)^\perp$. Now

$$\begin{aligned}[P_q, f]u_1 &= P_q f u_1 - f P_q u_1 \\ &= f u_1 - \bar\partial^* N_{q+1}\bar\partial(f u_1) - f u_1 \\ &= -\bar\partial^* N_{q+1}(\bar\partial f \wedge u_1).\end{aligned} \tag{4.1}$$

Since f has bounded derivatives, $u_1 \mapsto \bar\partial f \wedge u_1$ is bounded in $\mathcal{L}^2$, and because $\bar\partial^* N_{q+1}$ is compact, it follows that $[P_q, f]$ is compact on $\ker(\bar\partial)$. On the other hand,

$$\begin{aligned}[P_q, f]u_2 &= P_q f u_2 - f P_q u_2 \\ &= P_q f u_2 = P_q f \bar\partial^* N_{q+1}\bar\partial u_2 \\ &= P_q(f\bar\partial^* N_{q+1}\bar\partial u_2 - \bar\partial^* f N_{q+1}\bar\partial u_2);\end{aligned} \tag{4.2}$$

the last equality follows because P_q annihilates the range of $\bar\partial^*$. So we have

$$[P_q, f]u_2 = P_q[f, \bar\partial^*](N_{q+1}\bar\partial)u_2. \tag{4.3}$$

Because f has bounded derivatives, $[f, \bar\partial^*]$ acts as a bounded operator on $\mathcal{L}^2$, as does P_q. But $N_{q+1}\bar\partial = (\bar\partial^* N_{q+1})^*$ is compact, so that $[P_q, f]$ is compact on $\ker(\bar\partial)^\perp$.

Now assume only that f is in $C(\overline{\Omega})$. Extend f to a continuous function $\tilde f$ on $\mathbb{C}^n$ (for example, by the Tietze extension theorem). Approximate $\tilde f$ uniformly on $\overline{\Omega}$ by a sequence of functions $\{g_k\}_{k=1}^\infty \subseteq C^\infty(\mathbb{C}^n)$. Then the g_k (restricted to $\overline{\Omega}$) satisfy the additional assumption on the derivatives, so that the commutators $[P_q, g_k]$ are compact. It therefore suffices to see that these commutators converge to $[P_q, f]$ in operator norm. But

$$[P_q, f] - [P_q, g_k] = P_q(f - g_k) - (f - g_k)P_q. \tag{4.4}$$

Since the operator norm on $\mathcal{L}^2(\Omega)$ of multiplication by $(f - g_k)$ is $\max_{z\in\overline{\Omega}} |(f - g_k)|$, (4.4) implies that $\|[P_q, f] - [P_q, g_k]\| \to 0$ for $k \to \infty$. This completes the proof of Proposition 4.1. □

Remarks. (i) Proposition 4.1 is proved in [142] only under the additional assumption on the first derivatives of f made in the first part of the proof above. The authors thank S. Şahutŏglu for pointing out this oversight.

(ii) To what extent properties of commutators between the Bergman projection and multiplication operators, or between Toeplitz operators, can be used to characterize compactness properties in the $\bar\partial$-Neumann problem is open in general. However, compactness of the canonical solution operator $\bar\partial^* N_{q+1}$ restricted to forms with holomorphic coefficients is easily seen to be a consequence of compactness of the commutators $[P_q, \bar z_j]$, $1 \le j \le n$, see [266], Lemma 2, [142], Remark 2. Indeed, any $(0, q+1)$-form $u = \sum'_J u_J d\bar z_J$ can be written in the form $u = (1/(q+1)) \sum_j \sum'_K u_{j,K} d\bar z_j \wedge d\bar z_K$, and if u has holomorphic coefficients, the $u_{j,K}$ are holomorphic. Then we get from

(4.1) that

$$
\begin{aligned}
-\sum_j [P_q, \bar{z}_j]\Big(\sum_K{}' u_{j,K} d\bar{z}_K\Big) &= \sum_j \bar{\partial}^* N_{q+1}\Big(d\bar{z}_j \wedge \sum_K{}' u_{j,K} d\bar{z}_K\Big) \\
&= \bar{\partial}^* N_{q+1}\Big(\sum_j \sum_K{}' u_{j,K} d\bar{z}_j \wedge d\bar{z}_K\Big) \\
&= (q+1)\bar{\partial}^* N_{q+1} u.
\end{aligned}
\tag{4.5}
$$

For a formulation in terms of commutators of Toeplitz operators, see [184]. Remarkably, on convex domains, compactness of this restriction implies compactness of $\bar{\partial}^* N_{q+1}$ (even of N_{q+1}; [141], Remark (2) in Section 5; see also Remark (i) after the proof of Theorem 4.26 below). This provides, on convex domains, the characterization alluded to above.

4.2 General facts concerning compactness in the $\bar{\partial}$-Neumann problem

We now discuss some general facts concerning compactness in the $\bar{\partial}$-Neumann problem. We start with useful reformulations of the compactness property. As usual, $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ is provided with the graph norm.

Proposition 4.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n$. Then the following are equivalent:*

(i) *N_q is compact as an operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to itself;*

(ia) *N_q is compact as an operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$;*

(ii) *the embedding of $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ into $\mathcal{L}^2_{(0,q)}(\Omega)$ is compact;*

(iii) *for every $\varepsilon > 0$, there exists a constant C_ε such that we have the estimate*

$$\|u\|^2 \le \varepsilon(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2) + C_\varepsilon \|u\|^2_{-1} \text{ for } u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*);$$

(iv) *the canonical solution operators $\bar{\partial}^* N_q : \mathcal{L}^2_{(0,q)}(\Omega) \cap \ker(\bar{\partial}) \to \mathcal{L}^2_{(0,q-1)}(\Omega)$ and $\bar{\partial}^* N_{q+1} : \mathcal{L}^2_{(0,q+1)}(\Omega) \cap \ker(\bar{\partial}) \to \mathcal{L}^2_{(0,q)}(\Omega)$ are compact.*

The estimate in (iii) (actually, it is a family of estimates) is usually referred to as a compactness estimate. Note that the constant 1 in front of ε is immaterial, by rescaling of ε, any function $\sigma(\varepsilon)$ with $\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ will do. Saying that the canonical solution operator is compact is the same as saying that there exists *some* compact solution operator: compactness is preserved by projection onto the orthogonal complement of $\ker(\bar{\partial})$ (which produces the canonical solution). The proposition is essentially folklore, but see [201], Lemma 1.1 for the equivalence of (ii) and (iii), [221], Lemma 2.1, and [142], Lemma 1.1.

Proof of Proposition 4.2. Let $j_q : \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega)$ denote the canonical embedding where $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$ is provided with the graph norm. Recall from Theorem 2.9 that $N_q = j_q \circ j_q^*$ as an operator on $\mathcal{L}^2_{(0,q)}(\Omega)$, and $N_q = j_q^*$ as an operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$. Therefore, the claim that (i), (ia), and (ii) are equivalent amounts to the claim that $j_q \circ j_q^*$ compact $\Leftrightarrow$ j_q^* compact $\Leftrightarrow$ j_q compact. But an operator A is compact if and only if A^* is compact, and an operator of the form AA^* is compact if and only if both A and A^* are compact.

Note that in (iv), it is equivalent to say that $\bar\partial^* N_q$ and $\bar\partial^* N_{q+1}$ are compact on the full $\mathcal{L}^2$-spaces, since both vanish on the orthogonal complement of $\ker(\bar\partial)$. The equivalence of (i) and (iv) now follows from the formula

$$\begin{aligned} N_q &= N_q(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)N_q = (N_q\bar\partial)\bar\partial^* N_q + \bar\partial^* N_{q+1}(N_{q+1}\bar\partial) \\ &= (\bar\partial^* N_q)^*\bar\partial^* N_q + \bar\partial^* N_{q+1}(\bar\partial^* N_{q+1})^* \end{aligned} \tag{4.6}$$

(see (2.72)). We have commuted $\bar\partial^*$ and $\bar\partial$ with N_q in the second term, see (2.70); the parentheses are meant to indicate that we view $N_q\bar\partial$ and $N_{q+1}\bar\partial$ as bounded operators on all of $\mathcal{L}^2_{(0,q-1)}(\Omega)$ and $\mathcal{L}^2_{(0,q)}(\Omega)$, respectively (alternatively, one can observe that it suffices to establish (4.6) for smooth compactly supported forms, since such forms are dense in $\mathcal{L}^2_{(0,q)}(\Omega)$). Both operators on the right-hand side of (4.6) are nonnegative, so N_q is compact if and only if both those operators are compact. But again, an operator of the form A^*A is compact if and only if A is compact. This shows the equivalence of (i) and (iv).

The equivalence of (ii) with (iii) is a special case of Lemma 4.3 below, with $X = \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$ (with the graph norm), $Y = \mathcal{L}^2_{(0,q)}(\Omega)$, and T the inclusion $j_q : \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega)$. The implication (ii) $\Rightarrow$ (iii) follows from part (i) of Lemma 4.3, with $Z = W^{-1}_{(0,q)}(\Omega)$ and with S given by the composition of the inclusions $S\colon \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega) \hookrightarrow W^{-1}_{(0,q)}(\Omega)$. The reverse implication follows from part (ii) of the lemma with $Z_\varepsilon = W^{-1}_{(0,q)}(\Omega)$ and $S_\varepsilon = S$ for all $\varepsilon > 0$. Note that S is compact: by Rellich's lemma, the second inclusion in the definition of S is. This completes the proof of the proposition. □

The following lemma from functional analysis is very useful for dealing with compactness questions in our context. It appears in the literature in various forms, compare [201], Lemma 1.1, [215], Theorem 16.4, [93], Proposition V.2.3, [221], Lemma 2.1.

Lemma 4.3. *Assume X and Y are Hilbert spaces (over $\mathbb{C}$), $T\colon X \to Y$ is linear.*

(i) *Assume Z is a third Hilbert space, and $S\colon X \to Z$ is linear,* ***injective****, and continuous. If T is compact, then for all $\varepsilon > 0$ there is a constant C_ε such that*

$$\|Tx\|_Y \le \varepsilon\|x\|_X + C_\varepsilon\|Sx\|_Z. \tag{4.7}$$

(ii) *Assume that for all* $\varepsilon > 0$ *there are a Hilbert space* Z_ε*, a linear* ***compact*** *operator* $S_\varepsilon \colon X \to Z_\varepsilon$*, and a constant* C_ε *such that*

$$||Tx\|_Y \leq \varepsilon\|x\|_X + C_\varepsilon\|S_\varepsilon x\|_{Z_\varepsilon}. \tag{4.8}$$

Then T *is compact.*

Proof. To show (i), assume that the family of estimates (4.7) does not hold. Then, there is ε_0 and a sequence $\{x_n\}_{n=1}^\infty$ in X such that

$$\|Tx_n\|_Y > \varepsilon_0\|x_n\|_X + n\|Sx_n\|_Z. \tag{4.9}$$

In particular, $Tx_n \neq 0$, and we may rescale the x_n so that $\|Tx_n\|_Y = 1$. Then

$$1 > \varepsilon_0\|x_n\|_X + n\|Sx_n\|_Z. \tag{4.10}$$

By (4.10), $\{x_n\}_{n=1}^\infty$ is bounded in X. After passing to a subsequence, we may therefore assume that $\{x_n\}_{n=1}^\infty$ converges weakly to a limit, say x. Because T is compact, $\{Tx_n\}_{n=1}^\infty$ converges to Tx in Y. Because $\|Tx_n\|_Y = 1$ for all n, $\|Tx\|_Y = 1$. Also by (4.10), $Sx_n \to 0$ in Z. On the other hand, $Sx_n \to Sx$ weakly (the continuous operator S preserves weak convergence). Therefore, $Sx = 0$, whence $x = 0$ (S is injective). But then $Tx = 0$, which contradicts $\|Tx\|_Y = 1$. This contradiction establishes the family of estimates (4.7).

To prove (ii), assume (4.8) holds. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in X. It suffices to show that it contains a subsequence whose image under T is Cauchy in Y. Let $\varepsilon > 0$. Because S_ε is compact, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $S_\varepsilon x_{n_k}$ converges in Z_ε (and so in particular is Cauchy in Z_ε). Then (4.7) gives

$$\begin{aligned}\|Tx_{n_{k_1}} - Tx_{n_{k_2}}\|_Y &= \|T(x_{n_{k_1}} - x_{n_{k_2}})\|_Y \\ &\leq \varepsilon\|x_{n_{k_1}} - x_{n_{k_2}}\|_X + C_\varepsilon\|S_\varepsilon(x_{n_{k_1}} - x_{n_{k_2}})\|_{Z_\varepsilon}.\end{aligned} \tag{4.11}$$

Because $\{x_n\}_{n=1}^\infty$ is bounded in X and $\{S_\varepsilon x_{n_k}\}_{k=1}^\infty$ is Cauchy in Z_ε, the right-hand side of (4.11) can be made as small as we wish, say less than $1/2$, by first choosing ε small enough and then k_1 and k_2 big enough. This means that the sequence $\{Tx_{n_k}\}_{k=1}^\infty$ has the property that there exists K_0 such that $\|Tx_{n_m} - Tx_{n_k}\|_Y \leq 1/2$ for $m, k \geq K_0$. Repeating this procedure produces a subsequence of $\{Tx_{n_k}\}_{k=1}^\infty$ with an analogous property, but with $1/2$ replaced by $1/3$. Iteration now produces successive subsequences $\{Tx_k^j\}_{k=1}^\infty$ with the j-th one having the property that there exists an integer K_j so that $\|Tx_{k_1}^j - Tx_{k_2}^j\|_Y \leq 1/j$ for $k_1, k_2 \geq K_j$. This implies that the diagonal sequence $\{Tx_k^k\}_{k=1}^\infty$ is Cauchy, hence convergent, in Y. This completes the proof of Lemma 4.3. □

In contrast to global regularity (to be discussed in Chapter 4), compactness of the $\bar{\partial}$-Neumann operator can be localized:

Proposition 4.4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n$.*

(1) *If for every boundary point P there exists a pseudoconvex domain U such that $P \in U$ and $U \cap \Omega$ is a domain (i.e., connected) and N_q on $U \cap \Omega$ is compact, then N_q on Ω is compact.*

(2) *If U is a strictly pseudoconvex domain and $U \cap \Omega$ is a domain (i.e., connected), and if N_q is compact on Ω, then N_q is compact on $U \cap \Omega$.*

Proof. (1) follows from a partition of unity argument. Cover $b\Omega$ by finitely many of the open sets, say $U_1, U_2, \dots, U_m$. Let $\varphi_0, \varphi_1, \dots, \varphi_m$ be a partition of unity on $\bar{\Omega}$, such that supp $\varphi_0 \Subset \Omega$ and supp $\varphi_j \Subset U_j$, $1 \le j \le m$. Let $u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*) \subseteq \mathcal{L}^2_{(0,q)}(\Omega)$. Then

$$\|u\|^2 \lesssim \sum_{j=0}^{m} \|\varphi_j u\|^2 \le \|\varphi_0 u\|^2 + \varepsilon\Big(\sum_{j=1}^{m} \|\bar{\partial}(\varphi_j u)\|^2 + \|\bar{\partial}^*(\varphi_j u)\|^2\Big) + C_\varepsilon \sum_{j=1}^{m} \|\varphi_j u\|^2_{-1}. \tag{4.12}$$

We have used here that $\varphi_j u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ on $\Omega \cap U_j$. This is clear for $\mathrm{dom}(\bar{\partial})$; for $\mathrm{dom}(\bar{\partial}^*)$ it follows by pairing with $\bar{\partial}v$ for $v \in \mathrm{dom}(\bar{\partial})$ on $\Omega \cap U_j$, and then writing the integral as an integral over Ω. The term $\|\varphi_0 u\|^2$ can be estimated by interior elliptic regularity by $\varepsilon(\|\bar{\partial}(\varphi_0 u)\|^2 + \|\bar{\partial}^* \varphi_0 u\|^2)$. Expressing $\bar{\partial}(\varphi_j u)$ as $\bar{\partial}\varphi_j \wedge u + \varphi_j \bar{\partial} u$, and similarly for $\bar{\partial}^*(\varphi_j u)$, one can choose ε small enough so that the terms involving $\|u\|^2$ can be absorbed (the partition of unity is independent of ε). The result is the required compactness estimate on Ω.

The proof of part (2) requires some ideas which we will discuss in connection with Theorem 4.8 below. We therefore postpone this proof until after the proof of Theorem 4.8. □

Finally, we discuss an observation that, for smooth domains, is implicit in [197] and explicit in [253], [138], [222]: compactness percolates up the $\bar{\partial}$-complex.

Proposition 4.5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n-1$. If N_q is compact, then so is N_{q+1}.*

Proof. We will establish a compactness estimate on $(0, q+1)$-forms. Let $u = \sum'_{|J|=q+1} u_J d\bar{z}_J \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$. For $k = 1, \dots, n$, we define $(0,q)$-forms $v_k := \sum'_{|K|=q} u_{kK} d\bar{z}_K$. Then $v_k \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$. For $\mathrm{dom}(\bar{\partial})$, this holds because the components of $\bar{\partial}v_k$ are linear combinations of terms $\partial u_J/\partial\bar{z}_j$, and their $\mathcal{L}^2$-norm is controlled by $\|\bar{\partial}u\| + \|\bar{\partial}^* u\|$ (Corollary 2.13). To see that $v_k \in \mathrm{dom}(\bar{\partial}^*)$, note first that inner products with v_k are closely related to inner products with u: if

$\alpha = \sum'_{|K|=q} a_K d\bar{z}_K \in \mathcal{L}^2_{(0,q)}(\Omega)$, then for k fixed

$$\begin{aligned}(d\bar{z}_k \wedge \alpha, u) &= \Big(\sum_{|K|=q}{}' a_K (d\bar{z}_k \wedge d\bar{z}_K), \sum_{|J|=q+1}{}' u_J d\bar{z}_J)\Big) \\ &= \sum_{|K|=q}{}' a_K \overline{u_{kK}} = (\alpha, v_k).\end{aligned} \tag{4.13}$$

The inner products are in $\mathcal{L}^2_{(0,q+1)}(\Omega)$ and $\mathcal{L}^2_{(0,q)}(\Omega)$, respectively. Therefore, for $\beta \in \operatorname{dom}(\bar{\partial})$,

$$\begin{aligned}(\bar{\partial}\beta, v_k) &= (d\bar{z}_k \wedge \bar{\partial}\beta, u) = -(\bar{\partial}(d\bar{z}_k \wedge \beta), u) \\ &= -(d\bar{z}_k \wedge \beta, \bar{\partial}^* u) = -(\beta, \gamma_k),\end{aligned} \tag{4.14}$$

where $\gamma_k = \sum'_{|S|=q-1} (\bar{\partial}^* u)_{kS} d\bar{z}_S$. The last equality follows as in (4.13). (4.14) shows that $v_k \in \operatorname{dom}(\bar{\partial}^*)$, and that

$$\bar{\partial}^* v_k = -\gamma_k. \tag{4.15}$$

Now fix $\varepsilon > 0$. The compactness estimate for the q-forms v_k gives

$$\|u\|^2 = \frac{1}{q+1}\sum_{k=1}^{n}\|v_k\|^2 \le \frac{1}{q+1}\sum_{k=1}^{n}\Big(\varepsilon(\|\bar{\partial}v_k\|^2 + \|\bar{\partial}^* v_k\|^2) + C_\varepsilon \|v_k\|^2_{-1}\Big), \tag{4.16}$$

where the first equality follows from the definition of v_k and the observation that in the sum on the right-hand side of this equality, $\|u_J\|^2$ occurs precisely $(q+1)$ times for each strictly increasing multi-index J of length $q+1$. Both $\|\bar{\partial}v_k\|^2$ and $\|\bar{\partial}^* v_k\|^2$ are dominated by $\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2$, independently of ε. For $\|\bar{\partial}v_k\|^2$ this was noted at the beginning of the proof, for $\|\bar{\partial}^* v_k\|^2$, this follows from (4.15). Since also $\|v_k\|^2_{-1} \lesssim \|u\|^2_{-1}$, by the definition of v_k, (4.16) implies a compactness estimate for u. □

Remark. The proof of Proposition 4.5 can easily be adapted to show that subelliptic estimates percolate up the $\bar{\partial}$-complex as well ([253], [222]).

4.3 Estimates in Sobolev norms

From the point of view of the $\mathcal{L}^2$-Sobolev theory of the $\bar{\partial}$-Neumann problem, the most important application of compactness is that it implies global regularity. The following theorem comes from [201].

Theorem 4.6. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $1 \le q \le n$. Assume N_q is compact on $\mathcal{L}^2_{(0,q)}(\Omega)$. Then N_q is compact (in particular, continuous) from $W^s_{(0,q)}(\Omega)$ to itself for all $s \ge 0$.*

Of course, whether Theorem 4.6 provides a viable route to global regularity depends upon (the size of) the class of domains for which one can establish compactness. We will address this question after we prove the theorem.

Proof of Theorem 4.6. We will prove only the case when $s \in \mathbb{N}$; the general case may then be obtained from interpolation of linear operators (see [251] for the results concerning compactness). The proof follows [201]. As in the proof of Theorem 3.4, Chapter 3, we use elliptic regularization. Thus, if $u \in C^{\infty}_{(0,q)}(\overline{\Omega})$, $N_{\delta,q}u \in C^{\infty}_{(0,q)}(\overline{\Omega})$ for $\delta > 0$ sufficiently small. Fix a positive integer k. We want to show that for each $\varepsilon > 0$, there exists a constant C_ε (independent of $\delta > 0$) such that

$$\|N_{\delta,q}u\|_k \le \varepsilon\|u\|_k + C_\varepsilon\|u\|_0 \tag{4.17}$$

when $u \in C^{\infty}_{(0,q)}(\overline{\Omega})$. If (4.17) holds for $u \in C^{\infty}_{(0,q)}(\overline{\Omega})$, it holds for all $u \in W^k_{(0,q)}(\Omega)$, since $C^{\infty}_{(0,q)}(\overline{\Omega})$ is dense in $W^k_{(0,q)}(\Omega)$ (and $N_{\delta,q}$ is continuous in $\mathcal{L}^2_{(0,q)}(\Omega)$). We can then let δ tend to 0^+ (as in the proof of Theorem 3.4) to obtain the same estimate for $N_q u$:

$$\|N_q u\|_k \le \varepsilon\|u\|_k + C_\varepsilon\|u\|_0, \quad u \in W^k_{(0,q)}(\Omega). \tag{4.18}$$

(4.18) implies that N_q is compact; this follows from Lemma 4.3, part (ii), using that $W^k_{(0,q)}(\Omega) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega)$ is compact (since $k \ge 1$). Thus it suffices to establish (4.17) for $u \in C^{\infty}_{(0,q)}(\overline{\Omega})$.

In the following estimates, all constants will be uniform in $\delta > 0$. We first consider tangential derivatives of $N_{\delta,q}u$. Let $k \in \mathbb{N}$; we use the notation from the proof of Theorem 3.4 and let $T^\alpha = T_1^{\alpha_1}\dots T_{2n-1}^{\alpha_{2n-1}}$ denote a tangential operator of order k. Again, T_j is a smooth multiple (by a cutoff function) of $(\partial/\partial x_j)$, where $(x_1,\dots,x_{2n-1})$ are local coordinates in a special boundary chart; in particular, the T_j's are supported in a special boundary chart. We also let the T_j act componentwise in the special boundary chart, so that they preserve $\mathrm{dom}(\bar\partial^*)$. Then

$$\begin{aligned}\|T^\alpha N_{\delta,q}u\|^2 &\le \varepsilon(\|\bar\partial T^\alpha N_{\delta,q}u\|^2 + \|\bar\partial^* T^\alpha N_{\delta,q}u\|^2) + C_\varepsilon\|T^\alpha N_{\delta,q}u\|^2_{-1}\\ &\le \varepsilon Q_\delta(T^\alpha N_{\delta,q}u, T^\alpha N_{\delta,q}u) + C_\varepsilon\|N_{\delta,q}u\|^2_{k-1}\\ &\lesssim \varepsilon\left(|(T^\alpha u, T^\alpha N_{\delta,q}u)| + \|N_{\delta,q}u\|_k^2\right) + C_\varepsilon\|N_{\delta,q}u\|^2_{k-1}.\end{aligned} \tag{4.19}$$

The last inequality follows from inequality (3.50). Upon estimating $|(T^\alpha u, T^\alpha N_{\delta,q}u)|$ by $\|T^\alpha u\|^2 + \|T^\alpha N_{\delta,q}u\|^2$ and absorbing $\varepsilon\|T^\alpha N_{\delta,q}u\|^2$, this gives

$$\|T^\alpha N_{\delta,q}u\|^2 \lesssim \varepsilon(\|u\|_k^2 + \|N_{\delta,q}u\|_k^2) + C_\varepsilon\|N_{\delta,q}u\|^2_{k-1}. \tag{4.20}$$

We now address $\mathcal{L}^2$-norms of derivatives when some (possibly all) of the derivatives are normal. Let φ be a smooth cutoff function supported in a special boundary chart. It will be convenient to have the notation $|||\cdot|||_k$ for tangential k-norms, that is, $\mathcal{L}^2$-norms of tangential derivatives up to order k. Suppose a derivative of $\varphi N_{\delta,q}u$ of order k, when expressed in the coordinates of the boundary chart, contains precisely one normal derivative. Commuting both the normal derivative and the factor φ all the way to the left makes an error of order at most $k-1$. Therefore, such a term is estimated by

$$\|D^k\varphi N_{\delta,q}u\|^2 \lesssim \|\varphi(\partial/\partial\nu)T^\beta N_{\delta,q}u\|^2 + \|N_{\delta,q}u\|_{k-1}^2, \tag{4.21}$$

where $|\beta| = k-1$. $T^\beta N_{\delta,q}u$ is in the domain of $\bar\partial^*$. Lemma 2.2 and (3.50) give

$$\begin{aligned}
&\|\varphi(\partial/\partial\nu)T^\beta N_{\delta,q}u\|^2 \\
&\quad\lesssim |||T^\beta N_{\delta,q}u|||_1^2 + \|\bar\partial T^\beta N_{\delta,q}u\|^2 + \|\bar\partial^* T^\beta N_{\delta,q}u\|^2 + \|N_{\delta,q}u\|_{k-1}^2 \\
&\quad\lesssim |||\varphi N_{\delta,q}u|||_k^2 + Q_\delta(T^\beta N_{\delta,q}u, T^\beta N_{\delta,q}u) + \|N_{\delta,q}u\|_{k-1}^2 \\
&\quad\lesssim |||\varphi N_{\delta,q}u|||_k^2 + |(T^\beta u, T^\beta N_{\delta,q}u)| + \|N_{\delta,q}u\|_{k-1}^2 \\
&\quad\lesssim |||\varphi N_{\delta,q}u|||_k^2 + \|N_{\delta,q}u\|_{k-1}^2 + \|u\|_{k-1}^2.
\end{aligned} \tag{4.22}$$

Next, recall from (3.22) that

$$\Box_{\delta,q}u = -\left(\frac{1}{4}+\delta\right)\Delta u, \quad u \in \mathrm{dom}(\Box_{\delta,q}), \tag{4.23}$$

where Δ acts componentwise. Writing Δ in terms of normal derivatives and tangential derivatives (locally, in a boundary chart) as in (3.42), and differentiating $(m-2)$ times with respect to ν gives, for an integer m, $2 \le m \le k$, and a multi-index β with $|\beta| = k-m$:

$$\begin{aligned}
\varphi T^\beta \frac{\partial^{m-2}}{\partial\nu^{m-2}}\Delta N_{\delta,q}u &= \varphi T^\beta \frac{\partial^m}{\partial\nu^m}N_{\delta,q}u + \varphi D^{k-1}N_{\delta,q}u \\
&\quad + \varphi T^\beta\Bigl(\sum_{j,l=1}^{2n-1} a_{jl}\frac{\partial^m}{\partial\nu^{m-2}\partial t_k\partial t_j}N_{\delta,q}u\Bigr).
\end{aligned} \tag{4.24}$$

In the above equation, D^{k-1} denotes an expression involving at most $k-1$ derivatives, and the a_{jl} are smooth functions. Of course, the left-hand side of (4.24) is essentially $\varphi T^\beta\partial^{m-2}/(\partial\nu)^{m-2}u$ (from (4.23) above). Therefore, after commuting the tangential derivatives in the last expression in (4.24) to the left past the normal derivatives,

$$\left\|\varphi T^\beta \frac{\partial^m}{\partial\nu^m}N_{\delta,q}u\right\|^2 \lesssim \|u\|_{k-2}^2 + \|N_{\delta,q}u\|_{k-1}^2 + |||\varphi\frac{\partial^{m-2}}{\partial\nu^{m-2}}N_{\delta,q}u|||_{k-m+2}^2. \tag{4.25}$$

The last expression in (4.25) is of the same form as the expression on the left-hand side, modulo commuting φ with the tangential derivatives, which makes an error that is $O(\|N_{\delta,q}u\|_{k-1}^2)$. However, there are now only $(m-2)$ normal derivatives. Consequently, we can repeat this argument until there are no normal derivatives (m even), or until there is precisely one normal derivative (m odd). In the latter case, we invoke (4.22). The result (in both cases) is the estimate

$$\|\varphi N_{\delta,q}u\|_k^2 \lesssim |||\varphi N_{\delta,q}u|||_k^2 + \|N_{\delta,q}u\|_{k-1}^2 + \|u\|_{k-1}^2. \tag{4.26}$$

Let $\varphi_0, \varphi_1, \dots, \varphi_{j_0}$ be a partition of unity in a neighborhood of $\overline{\Omega}$, with $\varphi_0 \in C_0^\infty(\Omega)$, and φ_j compactly supported in a special boundary chart for $1 \leq j \leq j_0$. Applying (4.26) and then (4.20) to $\|\varphi N_{\delta,q} u\|_k^2$, and (2.84) (interior elliptic regularity) to $\|\varphi_0 N_{\delta,q} u\|_k^2$, we obtain

$$\begin{aligned}\|N_{\delta,q}u\|_k^2 &\lesssim \sum_{j=1}^{j_0} \|\varphi_j N_{\delta,q} u\|_k^2 + \|\varphi_0 N_{\delta,q} u\|_k^2 \\ &\lesssim \sum_{j=1}^{j_0} |||\varphi_j N_{\delta,q} u|||_k^2 + \|N_{\delta,q} u\|_{k-1}^2 + \|u\|_{k-1}^2 \\ &\lesssim \varepsilon(\|u\|_k^2 + \|N_{\delta,q}u\|_k^2) + C_\varepsilon \|N_{\delta,q}u\|_{k-1}^2 + \|u\|_{k-1}^2.\end{aligned} \tag{4.27}$$

Interpolating Sobolev norms for both u and $N_{\delta,q}u$ ($\|v\|_{k-1}^2 \leq \varepsilon'\|v\|_k^2 + C_{\varepsilon'}\|v\|_0^2$), using the continuity of $N_{\delta,q}$ in $\mathcal{L}^2_{(0,q)}(\Omega)$ with norm bounded independently of δ, and absorbing $\varepsilon\|N_{\delta,q}u\|_k^2$, gives (4.17). This completes the proof of Theorem 4.6. □

Remarks. (i) Note that Corollary 3.3 applies: $\bar\partial N_q$, $\bar\partial^* N_q$, $\bar\partial\bar\partial^* N_q$, and $\bar\partial^*\bar\partial N_q$ are all continuous in Sobolev norms when N_q is compact. So is P_{q-1}, see the remark after the proof of Corollary 3.3 (or go directly to Theorem 5.5).

(ii) The reader should note the following. To obtain Sobolev estimates in $W^k_{(0,q)}(\Omega)$ for k *fixed*, one does not need estimates (iii) in Lemma 4.2 for all $\varepsilon > 0$; indeed, the above proof shows that for a given domain Ω, there is $\varepsilon = \varepsilon(k) > 0$ so that estimates in $W^k_{(0,q)}(\Omega)$ follow if there is a constant C such that

$$\|u\|^2 \leq \varepsilon(k)(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) + C\|u\|_{-1}^2 \tag{4.28}$$

for $u \in \text{dom}(\bar\partial) \cap \text{dom}(\bar\partial^*)$. This observation has important implications; we shall return to this in Chapter 5 (Theorem 5.1).

4.4 A sufficient condition for compactness

The value of Theorem 4.6 depends on the size of the class of domains that satisfy compactness estimates. We now begin to address this question. Recall Corollary 2.13: if Ω is bounded and pseudoconvex, and if $b \in C^2(\overline{\Omega})$ is nonpositive, then

$$\sum_K{}' \sum_{j,k=1}^{n} \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar z_k} u_{jK}\overline{u_{kK}}\, dV \leq \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \tag{4.29}$$

for $u \in \text{dom}(\bar\partial) \cap \text{dom}(\bar\partial^*)$. Therefore, if there exist bounded such functions b with large Hessians, (4.29) should imply a compactness estimate. Moreover, by interior elliptic regularity, one only needs to estimate the norm of u in neighborhoods of the boundary.

These ideas can be made precise as follows. For a bounded pseudoconvex domain Ω in $\mathbb{C}^n$, we say that $b\Omega$ satisfies property (P_q) if the following holds: for every positive number M, there exists a neighborhood U of $b\Omega$ and a C^2 smooth function λ on U, such that $0 \leq \lambda(z) \leq 1$, $z \in U$, and such that for any $z \in U$, the sum of any q (equivalently: the smallest q) eigenvalues of the Hermitian form $\big(\frac{\partial^2\lambda}{\partial z_j \partial \bar{z}_k}(z)\big)_{jk}$ is at least M. This definition goes back to [68] (for $q = 1$, but the generalization to $q > 1$ is fairly straightforward; it may be found for example in [142]). More generally, we can replace $b\Omega$ in the definition of (P_q) by any compact subset $X \in \mathbb{C}^n$ (the terminology B-regular and B_q-regular is used in [272] and [142], respectively). Note that $(P_q) \Rightarrow (P_{q+1})$: if (P_q) holds, there can be at most $(q-1)$ negative eigenvalues, and adding the smallest $(q+1)$ eigenvalues adds a nonnegative term to the sum of the smallest q eigenvalues.

(P_q) says precisely that the sum on the left-hand side in (4.29) dominates $M\|u\|^2$; this follows from the following lemma from (multi)linear algebra. Denote by $\Lambda_z^{(0,q)}$ the space of $(0,q)$-forms at z (i.e., the space of skew symmetric q-linear functionals on $\mathbb{C}^n$).

Lemma 4.7. *Let λ be as above. Fix z and let $1 \leq q \leq n$. The following are equivalent:*

(i) *For all $u \in \Lambda_z^{(0,q)}$,*

$$\sideset{}{'}\sum_{|K|=q-1} \sum_{j,k=1}^{n} \frac{\partial^2\lambda(z)}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} \geq M|u|^2.$$

(ii) *The sum of any q (equivalently: the smallest q) eigenvalues of $\big(\frac{\partial^2\lambda(z)}{\partial z_j \partial \bar{z}_k}\big)_{j,k}$ is at least M.*

(iii) $\sum_{s=1}^{q} \sum_{j,k=1}^{n} \frac{\partial^2\lambda(z)}{\partial z_j \partial \bar{z}_k} (\underline{t}^s)_j \overline{(\underline{t}^s)_k} \geq M$ *whenever $\underline{t}^1, \underline{t}^2, \ldots, \underline{t}^q$ are orthonormal in $\mathbb{C}^n$.*

Proof. The equivalence of (i) and (ii) follows quickly when the Hermitian form $\big(\frac{\partial^2\lambda(z)}{\partial z_j \partial \bar{z}_k}\big)_{j,k}$ is diagonalized. Denote the eigenvalues by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then, in a suitable basis,

$$\begin{aligned}\sideset{}{'}\sum_{|K|=q-1} \sum_{j,k=1}^{n} \frac{\partial^2\lambda(z)}{\partial z_j \partial \bar{z}_k} u_{jk}\overline{u_{kK}} &= \sideset{}{'}\sum_{|K|=q-1} \sum_{j=1}^{n} \lambda_j |u_{jK}|^2 \\ &= \sideset{}{'}\sum_{J=(j_1,\ldots,j_q)} (\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_q})|u_J|^2.\end{aligned} \tag{4.30}$$

The last equality results as follows. For $J = (j_1, j_2, \ldots, j_q)$ fixed, $|u_J|^2$ occurs precisely q times in the second sum, once as $|u_{j_1K_1}|^2$, once as $|u_{j_2K_2}|^2$, etc. At each occurrence, it is multiplied by λ_{j_l}. Now clearly (ii) implies (i) and vice versa (fix $(j_1, j_2, \ldots, j_q)$ and set $u = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$).

We now show that (ii) and (iii) are equivalent. Assume (iii). Let $\underline{e}^j$ be an eigenvector associated with the eigenvalue λ_j. Then for any q-tuple $(j_1, \dots, j_q)$, $\underline{e}^{j_1}, \dots, \underline{e}^{j_q}$ are orthonormal, so that (iii) gives

$$\lambda_{j_1} + \cdots + \lambda_{j_q} = \sum_{s=1}^{q} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k} (\underline{e}^{j_s})_j \overline{(\underline{e}^{j_s})_k} \geq M. \tag{4.31}$$

Now assume (ii). Fix $\underline{t}^1, \dots, \underline{t}^q \in \mathbb{C}^n$. This set of vectors can be augmented to an orthonormal basis of $\mathbb{C}^n$ by $\underline{t}^{q+1}, \dots, \underline{t}^n$. Denote by (b_{jk}) the matrix of the Hermitian form $\sum_{j,k} (\partial^2 \lambda(z)/\partial z_j \partial \bar{z}_k) w_j \overline{w_k}$ in the basis $\underline{t}^1, \dots, \underline{t}^n$. Then the sum on the left-hand side of (iii) equals $b_{11} + \cdots + b_{qq}$. By the Schur majorization theorem (see for example [174], Theorem 4.3.26), this sum is no less than the sum of the smallest q eigenvalues, hence, by (ii), it is at least M. This completes the proof of the lemma. □

In view of its importance, it is convenient to have compact notation for the expression on the left-hand side of (i) in Lemma 4.7. We set

$$H_q(\lambda, z)(u, \bar{u}) := \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}}, \ u \in \Lambda_z^{(0,q)}. \tag{4.32}$$

The following theorem is from [68] for domains with sufficiently smooth boundaries; that no boundary regularity is required was shown in [283].

Theorem 4.8. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let $1 \leq q \leq n$. If $b\Omega$ satisfies property (P_q), then N_q is compact.*

Proof. Fix M, and choose λ_M according to the definition of property P_q: $\lambda_M \in C^2(U_M)$ for a neighborhood U_M of $b\Omega$, and the sum of any q eigenvalues of its complex Hessian is at least M. Denote by χ a smooth cutoff function that is compactly supported in U_M and is identically equal to 1 near $b\Omega$. Let $u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. Then (4.29) (i.e., Corollary 2.13) with $b = (\lambda_M - 1)$ gives

$$\begin{aligned}
&e^{-1} M \|u\|^2 \\
&\quad \leq e^{-1} M \|\chi u\|^2 + e^{-1} M \|(1-\chi)u\|^2 \\
&\quad \leq \sum_K{}' \sum_{j,k=1}^{n} \int_\Omega e^{(\lambda_M - 1)} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k} (\chi u)_{jK} \overline{(\chi u)_{kK}} \, dV + e^{-1} M \|(1-\chi)u\|^2 \\
&\quad \leq \|\bar{\partial}(\chi u)\|^2 + \|\bar{\partial}^*(\chi u)\|^2 + e^{-1} M \|(1-\chi)u\|^2 \\
&\quad \lesssim \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \|(\bar{\partial}\chi) \wedge u\|^2 + \sum_K{}' \left\| \frac{\partial \chi}{\partial z_j} u_{jK} \right\|^2 + e^{-1} M \|(1-\chi)u\|^2.
\end{aligned} \tag{4.33}$$

The last three terms in the last line of (4.33) are estimated by interior elliptic regularity, as follows. Choose a relatively compact subdomain V of Ω that contains the support of both $(1-\chi)$ and $\nabla\chi$. Then (see (2.85)), $\|u\|^2_{W^1(V)} \leq C_V(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2_{W^{-1}(\Omega)})$. Since $W_1(V) \hookrightarrow \mathcal{L}^2(V)$ is compact (Rellich's lemma), Lemma 4.3, part (i) gives that for every $\varepsilon > 0$, there exists a constant $C_{\varepsilon,V}$ such that

$$\|f\|^2_{\mathcal{L}^2(V)} \leq \varepsilon\|f\|^2_{W^1(V)} + C_{\varepsilon,V}\|f\|^2_{W^{-1}(V)},\ f \in W^1(V). \tag{4.34}$$

If ε is chosen small enough (depending on M), this shows that the last three terms in (4.33) can be dominated by $\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + C_M\|u\|^2_{W^{-1}(\Omega)}$. As a result, we obtain from (4.33)

$$\|u\|^2 \leq \frac{C}{M}(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) + C_M\|u\|^2_{-1}, \tag{4.35}$$

with a constant C that does not depend on M. Since M is arbitrary, (4.35) shows that N_q is compact (Proposition 4.2). □

Remarks. (i) In the above proof, we used Corollary 2.13 with $b = (\lambda_M - 1)$. The sesquilinear form $H_q(\lambda_M, z)$ in the second line of (4.33) is positive definite (in fact, large), so that Remark (i) following the proof of Corollary 2.13 applies. That is, we only need the functions λ_M from (P_q) on $U_M \cap \Omega$ (instead of on all of U); this gives a somewhat stronger result. We have chosen the stronger definition of (P_q) for two reasons. First, it turns out that on minimally smooth domains, for example when the boundary is locally a graph, the two versions are equivalent. This follows from Corollary 4.13 below ((P_q) is a local property of the boundary), and translation (locally, when the boundary is a graph, the weaker version implies the stronger one). The second reason is that the stronger version allows for an elegant study using methods of Choquet theory (see below).

(ii) In addition to only require the functions λ_M on $U_M \cap \Omega$, one can also relax the requirement that the λ_M's be C^2. For example, when $q = 1$, the complex Hessian of λ_M may be considered as a current when λ_M is merely a distribution, and the requirement then becomes that this current be at least $Mi\partial\bar{\partial}|z|^2$, in the sense of currents. For this, see [283], Corollary 3; for details about currents and their comparison, see for example [210], Chapter 2.

(iii) Theorems 4.6 and 4.8 together imply in particular, that if $b\Omega$ satisfies (P_1), then N_1 is continuous on $W^s_{(0,1)}(\Omega)$ for $s \geq 0$. If Ω is not assumed C^∞, but only C^k, then for $k \geq 2$, one can still conclude that N_1 is continuous on $W^{k-1}(\Omega)$, see [157], Theorem 1.1.

At this point, we pause to finish the proof of Proposition 4.4.

End of proof of Proposition 4.4. Let U be a strictly pseudoconvex domain so that $U \cap \Omega$ is connected. Assuming that the $\bar{\partial}$-Neumann operator is compact on $\mathcal{L}^2_{(0,q)}(\Omega)$, we have to show that the corresponding $\bar{\partial}$-Neumann operator $N_q^{U\cap\Omega}$ on $U \cap \Omega$ is compact. We show how to obtain a compactness estimate on $U \cap \Omega$. Fix $\varepsilon > 0$. Because the boundary of U is strictly pseudoconvex, it satisfies (P_1), hence (P_q) for all q. Using Corollary 2.13 as in (4.33) we find a suitable cutoff function φ_ε that is identically equal to 1 near bU and supported in a sufficiently small neighborhood of bU, so that for $u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*)$, we have the estimate (as in (4.33))

$$\begin{aligned}\|u\|^2 &\lesssim \|\varphi_\varepsilon u\|^2 + \|(1-\varphi_\varepsilon)u\|^2 \\ &\lesssim \varepsilon(\|\bar\partial(\varphi_\varepsilon u)\|^2 + \|\bar\partial^*(\varphi_\varepsilon u)\|^2) + \|(1-\varphi_\varepsilon)u\|^2 \\ &\lesssim \varepsilon(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) + C_\varepsilon\|\chi_\varepsilon u\|^2 + \|(1-\varphi_\varepsilon)u\|^2.\end{aligned} \tag{4.36}$$

Here, χ_ε is compactly supported in U and identically 1 on the support of $\nabla\varphi_\varepsilon$, and $C_\varepsilon = \max|\nabla\varphi_\varepsilon|$. We may view the forms $\chi_\varepsilon u$ and $(1-\varphi_\varepsilon)u$ as forms on Ω. Pairing with $\bar\partial v$, one checks that they are in $\mathrm{dom}(\bar\partial)\cap\mathrm{dom}(\bar\partial^*)$ on Ω. Applying the compactness estimate on Ω to these forms (with ε' to be chosen below) gives

$$\begin{aligned}\|u\|^2 \lesssim \varepsilon(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) &+ \varepsilon'\Big(C_\varepsilon(\|\bar\partial(\chi_\varepsilon u)\|^2 + \|\bar\partial^*(\chi_\varepsilon u)\|^2) \\ &\qquad + \|\bar\partial((1-\varphi_\varepsilon)u)\|^2 + \|\bar\partial^*((1-\varphi_\varepsilon)u)\|^2\Big) \\ &+ C_{\varepsilon'}(C_\varepsilon\|\chi_\varepsilon u\|^2_{-1} + \|(1-\varphi_\varepsilon)u\|^2_{-1}).\end{aligned} \tag{4.37}$$

The (-1)-norms in the last line of (4.37) are on Ω. However, because χ_ε and $(1-\varphi_\varepsilon)$ are compactly supported in U, these norms are dominated (with a constant depending on ε) by the (-1)-norms on $U\cap\Omega$: choose $\gamma_\varepsilon \in C_0^\infty(U)$ equal to 1 on the supports of χ_ε and $(1-\varphi_\varepsilon)$; γ_ε is a (continuous) multiplier from $W_0^1(\Omega)$ to $W_0^1(U\cap\Omega)$, hence from $W^{-1}(U\cap\Omega)$ to $W^{-1}(\Omega)$. But $\gamma_\varepsilon\chi_\varepsilon u = \chi_\varepsilon u$, and likewise for $(1-\varphi_\varepsilon)u$. Furthermore, the (-1)-norms in the last line of (4.37), taken on $U\cap\Omega$, are dominated by $\|u\|_{-1,U\cap\Omega}$. The terms in the second line of (4.37) give terms involving $\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$ and $\|u\|^2$. Choosing ε' small enough results, after absorbing the $\|u\|^2$ term, in the desired compactness estimate

$$\|u\|^2 \lesssim \varepsilon(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) + \widetilde{C}_\varepsilon\|u\|^2_{-1}, \tag{4.38}$$

where all the norms are on $U\cap\Omega$. This completes the proof of Proposition 4.4. □

4.5 Analysis of property (P_q)

We proceed to study property (P_q). In this, we follow Sibony [272] (for $q = 1$, but his arguments carry over to the general case, see [142]), who realized that this notion fits neatly within the framework of Choquet theory for the cone of plurisubharmonic functions (see Chapter 1 in [145] for an exposition of Choquet theory in this context, [252] for the general theory). Actually, when $q > 1$, the appropriate class of functions consists of the functions that are continuous in an open set U and subharmonic on (the intersection with U of) every q-dimensional affine subspace, with the metric induced by the ambient $\mathbb{C}^n$. We denote this class by $P_q(U)$. The remark about the metric is needed because while harmonicity is invariant under unitary linear transformations, it is not under general affine linear transformations. So coordinates in the affine subspace are to be taken with respect to a basis that is orthonormal in the inner product induced by $\mathbb{C}^n$. In particular,being a member of $P_q(U)$ for some open set U is not preserved under biholomorphic maps when $q > 1$, since these maps are in general not conformal. As a consequence, whether or not (P_q) is invariant also when $q > 1$ is not clear. Nonetheless, property (P_q) turns out to be a useful notion in these cases also.

Lemma 4.7 immediately gives the following characterization of the functions in $C^2(U) \cap P_q(U)$ (see also [175], where functions in $P_q(U)$ are studied under the name 'q-subharmonic functions'):

Lemma 4.9. *Let U be an open subset of $\mathbb{C}^n$, $\lambda \in C^2(U)$. Then $\lambda \in P_q(U)$ if and only if at every point of U, the sum of any q (equivalently: the smallest q) eigenvalues of its complex Hessian is nonnegative.*

Proof of Lemma 4.9. If $(\zeta_1, \dots, \zeta_q)$ are coordinates in the affine subspace with respect to the orthonormal basis $\underline{t}_1, \dots, \underline{t}_q$, then the left-hand side of (iii) in Lemma 4.7 equals $\sum_{s=1}^{q}(\partial^2\lambda/\partial\zeta_s\partial\overline{\zeta_s}) = (1/4)\Delta\lambda$. The corollary now follows from the equivalence of (ii) and (iii) in Lemma 4.7 (for $M = 0$). □

Denote by $P_q(X)$ the closure in the algebra $C(X)$ of continuous functions on X of the functions that belong to $P_q(U)$ for an open neighborhood U of X (U is allowed to depend on the function). A probability measure μ on X is a q-Jensen measure for $z \in X$ if

$$h(z) \le \int_X h\, d\mu \quad \text{for all } h \in P_q(X). \tag{4.39}$$

That is, the functions in $P_q(X)$ subaverage at z with respect to μ. In the terminology of [145], Chapter 1, these measures are the R-measures for the family $R = P_q(X)$. Denote by $J_q(X)$ the associated Choquet boundary: the set of points $z \in X$ for which the only q-Jensen measure is the unit point mass at z.

Proposition 4.10. *Let X be a compact subset of $\mathbb{C}^n$. The following are equivalent:*

(i) *X satisfies property (P_q),*

(ii) $P_q(X) = C(X)$,

(iii) $J_q(X) = X$,

(iv) *the function $-|z|^2$ belongs to $P_q(X)$.*

Proof. The equivalence of (i), (ii), and (iv) is a relatively easy consequence of the definitions. To see that (i) implies (ii), let $f \in C(X)$. We may assume that $f \in C^2$ in a neighborhood of X (these functions are dense in $C(X)$). (i) implies that for all $t > 0$, there is a C^2-function λ_t, $0 \le \lambda_t \le 1$, in a neighborhood of X such that $tf + \lambda_t \in P_q(U_t)$ for some neighborhood U_t of X. But then $f + (1/t)\lambda_t \in P_q(U_t)$, and $f + (1/t)\lambda_t \to f$, uniformly on X. (ii) trivially implies (iv). When (iv) holds, and $M > 0$, then $-M|z|^2 \in P_q(X)$, and there is $\lambda \in P_q(U)$, U a neighborhood of X, with $-(1/2) \le \lambda - (-M|z|^2) \le 1/2$, i.e., $0 \le \lambda + M|z|^2 + (1/2) \le 1$. Functions in $P_q(U)$ can be approximated by smooth functions in $P_q(U)$ in the same way that plurisubharmonic functions (that is, functions in $P_1(U)$) can be approximated by smooth plurisubharmonic functions. We may therefore assume that λ is smooth. Then $\lambda + M|z|^2 + (1/2)$ is smooth, and the sum of the smallest q eigenvalues of its complex Hessian is at least M (the Laplacian on any q-dimensional affine subspace is at least M). Since $M > 0$ was arbitrary, we have established (i).

That (ii) implies (iii) is clear: (ii) implies that the subaveraging property (4.39) of q-Jensen measures holds for continuous functions. Therefore, (iii) holds. To prove that (iii) implies (ii), we need a result of Edwards (see [145], Theorem 1.2) that says that the following two quantities are equal: for $f \in C(X)$, we have

$$\sup\{\lambda(z) \mid \lambda \in P_q(X), \lambda \le f \text{ on } X\} = \inf\left\{\int_X f\, d\mu \mid \mu \in J_{q,z}\right\}, \tag{4.40}$$

where $J_{q,z}$ denotes the set of q-Jensen measures for z. Note that if $\lambda \in P_q(X)$ and $\lambda \le f$ on X, then $\lambda(z) \le \int_X \lambda\, d\mu \le \int_X f\, d\mu$, so that the left-hand side of (4.40) is trivially less than or equal to the right-hand side. The other, nontrivial, direction is a consequence of a geometric Hahn–Banach type theorem about separation of convex sets. When (iii) holds, the right-hand side of (4.40) equals $f(z)$, and thus $f(z) = \sup\{\lambda(z) \mid \lambda \in P_q(X), \lambda \le f \text{ on } X\}$. It now follows from compactness of X that f is the uniform limit of functions of the form $\max\{\lambda_1, \dots, \lambda_k\}$, where all λ_j are in $P_q(U)$ for some neighborhood U of X. But such a maximum belongs itself to $P_q(U)$ (subharmonicity on q-dimensional affine subspaces is preserved), whence $f \in P_q(X)$. We have shown that (iii) implies (ii). This concludes the proof of Proposition 4.10. □

Remark. One immediate consequence of Proposition 4.10 is that analytic discs in the boundary are obstructions to property (P_1) (in view of (ii) and the maximum principle for subharmonic functions, or in view of (iii)). It is natural to ask whether they are obstructions to compactness as well. We will have more to say on this subject later (see Theorem 4.21).

Abstract Choquet theory provides the following useful characterization of points in $J_q(X)$ ([145], Theorem 1.13).

Proposition 4.11. *Let $X \subset \mathbb{C}^n$ be compact. For a point $z \in X$, the following are equivalent:*

(i) *$z \in J_q(X)$,*

(ii) *for every $f \in C(X)$, there exists $\lambda \in P_q(X)$ with $\lambda(z) = f(z)$ and $\lambda \le f$ on X,*

(iii) *there exist constants $\alpha < 0 < \beta$ such that for every compact subset E of X not containing z, there is a function $\lambda \in P_q(X)$ that satisfies $\lambda(z) = 0$, $\lambda \le \alpha$ on E, and $\lambda \le \beta$ on X.*

With the help of this characterization of $J_q(X)$, it is easy to see that property (P_q) is a local property. In fact, more can be said. For $z \in \mathbb{C}^n$ and $r > 0$, denote by $B(z,r)$ the (open) ball of radius r centered at z.

Lemma 4.12. *Let $X \subset \mathbb{C}^n$ be compact, $z_0 \in X$, and $r > 0$. Then*

$$J_q(X \cap \overline{B(z_0,r)}) \setminus bB(z_0,r) \subseteq J_q(X).$$

It is not hard to see that the exclusion of $bB(z_0,r)$ is necessary. For example, let $X \subset \mathbb{C}^2$, $X = \{(0,0)\} \cup \{(1,w)||w| \leq 1\}$. Let $z_0 = (0,0)$, $r = 1$. Then $X \cap \overline{B(z_0,r)} = \{(0,0),(1,0)\}$, and $(1,0) \in J_1(X \cap \overline{B(z_0,r)})$ (since any finite set satisfies property (P_1)). But $(1,0) \notin J_1(X)$, as it is the center of an analytic disc in X.

Proof of Lemma 4.12. Let $z \in J_q(X \cap \overline{B(z_0,r)}) \setminus bB(z_0,r)$, and let E be a compact subset of X not containing z. Let $\alpha < 0 < \beta$ be the constants from (iii) in Proposition 4.11 (applied to $z \in J_q(X \cap \overline{B(z_0,r)})$). Set $\tilde{E} = (E \cap \overline{B(z_0,r)}) \cup (X \cap bB(z_0,r))$. $\tilde{E}$ is a compact subset of $X \cap \overline{B(z_0,r)}$ that does not contain z. By Proposition 4.11, there is a function $\tilde{v} \in P_q(X \cap \overline{B(z_0,r)})$ with $\tilde{v} < \alpha$ on $\tilde{E}$, $\tilde{v}(z) = 0$, and $\tilde{v} < \beta$ on $X \cap \overline{B(z_0,r)}$. By approximation followed by translation, we may assume that $\tilde{v} \in P_q(U)$ for an open neighborhood U of $X \cap \overline{B(z_0,r)}$. Set $v(\zeta) = \max\{\tilde{v}(\zeta), \alpha/2\}$ on $U \cap B(z_0,r)$, and $v = \alpha/2$ on the complement of $B(z_0,r)$. It is easy to check that $v \in P_q(V)$ for some open neighborhood V of X, and $v < \alpha$ on E, $v(z) = 0$, and $v < \beta$ on X. We have verified (iii) in Proposition 4.11. Therefore, $z \in J(X)$. □

Corollary 4.13. *Let X be a compact subset of $\mathbb{C}^n$. Assume that for every $z \in X$, there exists $r > 0$ such that $X \cap \overline{B(z,r)}$ satisfies property (P_q). Then so does X.*

Proof. Let $z \in X$, and $r > 0$ such that $X \cap \overline{B(z,r)}$ satisfies property (P_q). By Proposition 4.10 and Lemma 4.12, $z \in J_q(X \cap \overline{B(z,r)}) \setminus bB(z,r) \subseteq J_q(X)$. It follows that $J_q(X) = X$, and X satisfies (P_q) (again by Proposition 4.10). □

Lemma 4.12 also implies that Property (P_q) is preserved under countable unions, as follows.

Corollary 4.14. *Let $X = \bigcup_{k=1}^{\infty} X_k$, with X_k compact for all k. Assume that X is compact. If all X_k satisfy property(P_q), then so does X.*

Proof. By (4.40), it suffices to show that when $f \in C(X)$, then $\sup\{\lambda(z)|\lambda \in P_q(X), \lambda \leq f \text{ on } X\} = f(z)$. So fix f. The set A where the preceding equality fails (i.e., the left-hand side is less than $f(z)$) is open in X. Assume A is not empty. A is locally compact, so by the Baire category theorem, there is k_0 such that $A \cap X_{k_0}$ has nonempty interior in A, hence in X. Pick a point z_0 in this interior. Thus there is $r > 0$ such that $X \cap B(z_0,r) \subseteq A \cap X_{k_0}$. But then $X \cap \overline{B(z_0,r/2)} \subseteq X_{k_0}$. Consequently, $X \cap \overline{B(z_0,r/2)}$ satisfies (P_q) as well (that (P_q) passes to compact subsets is obvious from the definition). Therefore, $z_0 \in J(X \cap \overline{B(z_0,r/2)}) \setminus bB(z_0,r/2) \subseteq J(X)$. The last inclusion is from Lemma 4.12. Because $z_0 \in J(X)$, (4.40) implies $\sup\{\lambda(z_0)|\lambda \in P_q(X), \lambda \leq f \text{ on } X\} = f(z_0)$. This contradicts the fact that $z_0 \in A$. That is, the assumption that A is nonempty leads to a contradiction. □

There is more information in [272]. In particular, when Ω is regular enough, (P_q) can also be characterized (in addition to the properties given in Proposition 4.10) by the existence of peak functions in $P_q(\Omega) \cap C(\overline{\Omega})$, or by the fact that every continuous function on $b\Omega$ is the boundary value of a function in $P_q(\Omega) \cap C(\overline{\Omega})$ ([272], Théorème 2.1 for the case $q = 1$, but the arguments carry over to general q).

Remark. On Lipschitz domains, property (P_1) can be reformulated in the spirit of Oka's lemma: $b\Omega$ satisfies (P_1) if and only if it locally admits functions ρ comparable to minus the boundary distance so that the complex Hessian of $-\log(-\rho)$ tends to infinity upon approach to the boundary ([156]).

4.6 Some examples

The simplest examples of domains that satisfy (P_1) (hence (P_q) for all q) are strictly pseudoconvex domains. Such domains admit a *strictly* plurisubharmonic defining function (see the proof of Theorem 3.7 and the reference there), and (P_1) follows immediately. Of course, we already know from Chapter 3 that the $\bar{\partial}$-Neumann problem on these domains is subelliptic (with a gain of one derivative for N_q); this is considerably stronger than compactness. More generally, when the domain is of finite 1-type (see the discussion in Chapter 3, and the references given there), then the boundary satisfies property (P_1) (hence (P_q) for all q). In fact, this was the motivation for introducing this notion in [68]. Of course, Catlin later proved subellipticity for such domains ([70]). The point in [68] was that the weakly pseudoconvex boundary points are contained in a union of certain special submanifolds of the boundary. Let S be a smooth submanifold of the boundary $b\Omega$. For $P \in S$, denote by $T_P^{\mathbb{C}}(S)$ the complex tangent space to S at P, and by $\mathcal{N}_P$ the null space of the Levi form of $b\Omega$ at P. The following proposition is implicit in [68] and explicit in [272] (for $q = 1$).

Proposition 4.15. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $1 \le q \le n$, and assume S is a smooth submanifold of $b\Omega$ such that $\dim_{\mathbb{C}}(T_P^{\mathbb{C}}(S) \cap \mathcal{N}_P) < q$ for all $P \in S$. Then any compact subset of S satisfies property (P_q).*

Remark. The quantity $\dim_{\mathbb{C}}(T_P^{\mathbb{C}}(S) \cap \mathcal{N}_P)$ is important for the notion of 'holomorphic dimension' introduced by Kohn ([193]) in his work on subelliptic multipliers; see also [107] and [18].

Note that totally real submanifolds, that is, submanifolds without complex tangents, trivially satisfy the assumption in Proposition 4.15, for all q. On the other hand, when S is a complex submanifold of $b\Omega$ of dimension m, then $\dim_{\mathbb{C}}(T_P^{\mathbb{C}}(S) \cap \mathcal{N}_P) = m$ for all $P \in S$.

We can now give examples of domains where subellipticity fails (i.e., which are not of finite type), but where compactness holds. This is illustrated by the following corollary.

Corollary 4.16. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded pseudoconvex domain, $1 \le q \le n$. Assume that the boundary points of infinite type are contained in a submanifold of the boundary that satisfies the assumptions in Proposition 4.15. Then N_q is compact.*

Proof. The set K of points of infinite type is a compact subset of $b\Omega$ ([89], [92]). By Proposition 4.15, it satisfies property (P_q). $b\Omega \setminus K$ can be written as a countable union of compact subsets of itself. Using the results from [68] (specifically, a local version of Theorem 2), we obtain that these compact subsets also satisfy (P_q). Therefore, $b\Omega$ is the countable union of compact sets all of which satisfy (P_q). By Corollary 4.14, $b\Omega$ satisfies (P_q) as well, and Theorem 4.8 applies. □

Remarks. (i) The previous argument shows that the assumption in Theorem 4.8 can be modified: it suffices to require that the set K of boundary points of infinite type (1-type in the sense of [89]) satisfy (P_q). Then $b\Omega$ satisfies (P_q). The conclusion that if the set of points of infinite type satisfies (P_q), then N_q is compact can also be obtained by exploiting the pseudolocal estimates near points of finite type (see Chapter 2) to modify the proof of Theorem 4.8.

(ii) Let $q = 1$. If the points of infinite type are contained in a smooth submanifold S of the boundary of (real) dimension less than two, then the assumptions in Corollary 4.16 are satisfied (and consequently N_1 is compact). There is a 'nonsmooth' version of this fact, formulated in terms of two-dimensional Hausdorff measure: if the points of infinite type are contained in a set of two-dimensional Hausdorff measure zero, then $b\Omega$ satisfies (P_1); see [272], remarque on page 310, and [42], where an explicit construction of the required plurisubharmonic functions is given.

Proof of Proposition 4.15. We still follow [272]. Let S as in the assumption, $K \subseteq S$, K compact. It suffices to show that there exists a C^2 function λ in a neighborhood U of K such that $\lambda = 0$ on K, and such that there is a constant $a > 0$ so that at any point of K, the sum of the smallest q eigenvalues of the complex Hessian of λ is at least a. Pick a defining function ρ for Ω. In view of Corollary 4.13, we may assume that near K, the submanifold S is given by $\rho = \rho_1 = \cdots = \rho_m = 0$ for suitable smooth functions $\rho_1, \ldots, \rho_m$. Set

$$\lambda(z) = \rho(z) + C\Big(\rho(z)^2 + \sum_{l=1}^{m} \rho_l(z)^2\Big), \tag{4.41}$$

where the positive constant C is to be determined momentarily. Then $\lambda = 0$ on K. To check the second property, we verify (iii) in Lemma 4.7. For an orthonormal set of vectors $\underline{t}^1, \ldots, \underline{t}^q$, we have for $z \in K$

$$\begin{aligned} &\sum_{s=1}^{q} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k} (\underline{t}^s)_j \overline{(\underline{t}^s)_k} \\ &\quad = \sum_{s=1}^{q} \sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} (\underline{t}^s)_j \overline{(\underline{t}^s)_k} \\ &\qquad + 2C \sum_{s=1}^{q} \Big(\Big| \sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j} (\underline{t}^s)_j \Big|^2 + \sum_{l=1}^{m} \Big| \sum_{j=1}^{n} \frac{\partial \rho_l(z)}{\partial z_j} (\underline{t}^s)_j \Big|^2 \Big). \end{aligned} \tag{4.42}$$

As long as $C \geq 0$, the right-hand side of (4.42) is strictly positive when the first term is. On the other hand, when $\sum_{s=1}^{q}\sum_{j,k=1}^{n}\frac{\partial^2\rho(z)}{\partial z_j\partial\bar{z}_k}(\underline{t}^s)_j\overline{(\underline{t}^s)_k} \leq 0$, the second line in (4.42) is positive. Otherwise, all $\underline{t}^s$ would be in the complex tangent space to S at z as well as in the null space of the Levi form at z, contradicting the assumption on S.(We have used here that if all $\underline{t}^s$ are complex tangential to $b\Omega$, then $\sum_{j,k=1}^{n}\frac{\partial^2\rho(z)}{\partial z_j\partial\bar{z}_k}(\underline{t}^s)_j\overline{(\underline{t}^s)_k} \geq 0$ for all s, by pseudoconvexity.) Consequently, C can be chosen big enough to make the right-hand side of (4.42) (strictly) positive. By compactness, this can be done uniformly for $z \in K$, and the proof is complete. □

In the examples furnished by Proposition 4.15 (as well as the ones from Remark (ii) above) the sets of boundary points of infinite type have (surface) measure zero. However, there are domains whose boundary satisfies (P_1), yet whose boundary points of infinite type form a subset of the boundary of positive measure. The construction of the examples rests on a suitable characterization of compact sets in the plane that satisfy (P_1) (the equivalence of (1) and (2) in Proposition 4.17 below). Recall that the fine topology in an open subset U of the complex plane is the coarsest topology that makes all subharmonic functions in U continuous. For properties of this topology, we refer the reader to [166], Chapter 10, or [3], Chapter 7. In particular, the fine topology on an open set is strictly finer than the Euclidean topology, and the restriction of the fine topology of $\mathbb{C}$ to an open subset U coincides with the fine topology of U. Finely open sets are still 'massive' near a point: if z is a fine interior point of a set $E \subseteq \mathbb{C}$, then $\lim_{r\to 0}(\sigma(bB(z,r)\cap E)/2\pi r) = 1$, where σ denotes arclength and $B(z,r)$ is as usual the ball (i.e., disc) centered at z with radius r ([166], Corollary 10.5; [3], Corollary 7.2.4). Moreover, there exist arbitrarily small positive r such that $bB(z,r) \subseteq E$ ([166], Theorem 10.14; [3], Theorem 7.3.9).

The following proposition combines work from [272] and [135]. It is formulated in [143]. For an open subset U of $\mathbb{C}$, denote by $\lambda_1(U)$ the first eigenvalue of the Dirichlet problem for the Laplacian on U, i.e., $\lambda_1(U) = \inf\{\int|\nabla u|^2 \mid u \in C_0^\infty(U), \int|u|^2 = 1\}$ (compare e.g. [77]).

Proposition 4.17. *Let K be a compact subset of $\mathbb{C}$. The following are equivalent:*

(1) *K satisfies property (P_1).*

(2) *K has empty fine interior.*

(3) *K supports no nonzero function in $W_0^1(\mathbb{C})$.*

(4) *For any sequence of open sets $\{U_j\}_{j=1}^{\infty}$ such that $K \subset\subset U_{j+1} \subset\subset U_j$ and $\bigcap_{j=1}^{\infty}\overline{U_j} = K$, $\lambda_1(U_j) \to \infty$ as $j \to \infty$.*

By what was said above, the statement in (ii) may be taken with respect to the fine topology on any open subset U of $\mathbb{C}$ that contains K.

Proof. We first show the equivalence of (1) and (2) (this equivalence is from [272]). Assume there is a fine interior point $z_0 \in K$. Then $\sigma(bB(z_0,r)\cap K)/2\pi r \to 1$ (see the discussion above). (iii) in Proposition 4.11 now shows that $z_0 \notin J(K)$. To see this, take a circle centered at z_0 of small enough radius as the compact set E, and use subaveraging; note that α and β are independent of E, hence of the radius of the circle. We have shown: if the fine interior of K is not empty, then K does not satisfy (P_1). On the other hand, assume that K has empty fine interior, and let $z_0 \in K$. Then z_0 is a fine boundary point of K. If μ is a Jensen measure for z_0, then in particular the restrictions to K of functions harmonic in some neighborhood of K have to subaverage with respect to μ at z_0. That is, μ is an R-measure for z_0, in the terminology of [145], associated to the family R of functions that are harmonic in some neighborhood of K. Theorem X.3 in [57] says that the corresponding Choquet boundary equals the fine boundary of K. (The theorem requires that K be a compact subset of a 'Green space', which $\mathbb{C}$ is not. But any bounded open subset U is, and since the fine interior of K with respect to such a subset U is the same as the fine interior with respect to $\mathbb{C}$, the conclusion remains valid in our case.) Since z_0 is a fine boundary point of K, it belongs to this Choquet boundary, and consequently, μ is the point unit mass at z_0. In other words, $z_0 \in J_1(K)$. Since $z_0 \in K$ was arbitrary, this shows that $K = J_1(K)$, i.e., K satisfies (P_1), by Proposition 4.10. We have shown that (1) and (2) are equivalent.

It was observed in [143] that the equivalence of (2), (3), and (4) is essentially in [135]. The Dirichlet problem for the Laplacian, familiar in the setting of open sets, can also be formulated for finely open sets ([135] and the references there). The resulting theory inherits many features of the classical theory, but avoids some of its problems related to stability of eigenvalues under unions or intersections of sequences of domains. We continue to use the notation $\lambda_1(U)$ and $W_0^1(U)$ when U is only assumed finely open. If (2) holds then $\{0\} = W_0^1(\mathrm{int}_f K) = W_0^1(K)$, where $int_f K$ denotes the fine interior of K. For the second equality, see [135], equation (3) on p. 93. But $W_0^1(K) = \{0\}$ implies (3). Assume that (3) holds. Then (4) must hold. Otherwise, there would exist a sequence of functions $\{u_j\}_{j=1}^{\infty}$, $u_j \in W_0^1(U_j)$, with $\|u_j\| = 1$ and $\|\nabla u_j\| \le C$, for a suitable sequence $\{U_j\}_{j=1}^{\infty}$ and a constant C. Passing to a subsequence that converges both weakly in $W_0^1(\mathbb{C})$ and in $\mathcal{L}^2$ of a neighborhood of K yields a nonzero element of $W_0^1(\mathbb{C})$ that is supported on K, contradicting (3). Finally, that (4) implies (2) is a consequence of Theorem 2, part 1° in [135]: choose a sequence $\{U_j\}_{j=1}^{\infty}$ as in (4); then $\lim_{j\to\infty} \lambda_1(U_j) = \lambda_1(\mathrm{int}_f \bigcap_j U_j) = \lambda_1(\mathrm{int}_f K)$. This forces $\mathrm{int}_f K$ to be empty, otherwise, the last quantity would be finite. This completes the proof of Proposition 4.17. □

4.7 Hartogs domains in $\mathbb{C}^2$

Hartogs domains in $\mathbb{C}^2$ are domains with rotational symmetry in the second variable (see e.g. [259] for properties of these domains). Many interesting examples and counterexamples in several complex variables are found within this class of domains ([105], [183], [9], [11], [83], [49], [130], [205], [213]). We are going to discuss examples of Hartogs domains from [272] whose boundary points of infinite type form a set of positive measure, yet whose $\bar{\partial}$-Neumann operator is compact. There will also be examples of Hartogs domains (also from [272]) which fail property (P) although their boundaries contain no analytic discs. This requires some preparation. Suppose Ω is a smooth complete Hartogs domain over the unit disc $\mathbb{D}$: $\Omega = \{(z,w) \mid |w|^2 < e^{-\varphi(z)}\}$, with $\varphi \in C^\infty(\mathbb{D})$. Ω is pseudoconvex if and only if φ is subharmonic. Over $\mathbb{D}$, we may take $\rho(z,w) = |w|^2 - e^{-\varphi(z)}$ as a defining function. We have

$$\begin{gathered}\frac{\partial^2\rho}{\partial z\partial\bar{z}}(z,w) = -e^{\varphi(z)}\left|\frac{\partial\varphi}{\partial z}\right|^2 + e^{-\varphi(z)}\frac{\partial^2\varphi}{\partial z\partial\bar{z}},\\ \frac{\partial^2\rho}{\partial w\partial\bar{w}}(z,w) = 1 \quad \text{and} \quad \frac{\partial^2\rho}{\partial z\partial\bar{w}}(z,w) = \frac{\partial^2\rho}{\partial\bar{z}\partial w}(z,w) = 0.\end{gathered} \tag{4.43}$$

The complex tangent space $T^{\mathbb{C}}_{(z,w)}(b\Omega)$ at a point $(z,w) \in b\Omega$ is given by $\{(\zeta_1,\zeta_2) \mid e^{-\varphi(z)}(\partial\varphi/\partial z)(z)\zeta_1 + \bar{w}\zeta_2 = 0\}$, and so is spanned by $(1, -(e^{-\varphi(z)}/\bar{w})\,(\partial\varphi/\partial z)(z))$. Inserting this vector into the Levi form results in (in view of (4.43) and the fact that $|w|^2 = e^{-\varphi}$ on the boundary)

$$-e^{-\varphi}\left(\left|\frac{\partial\varphi}{\partial z}\right|^2 - \frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right) + \frac{e^{-2\varphi}}{|w|^2}\left|\frac{\partial\varphi}{\partial z}\right|^2 = e^{-\varphi}\frac{\partial^2\varphi}{\partial z\partial\bar{z}} = \frac{1}{4}e^{-\varphi}\Delta\varphi. \tag{4.44}$$

Consequently, a boundary point (z,w) with $z \in \mathbb{D}$ is a weakly pseudoconvex point precisely when $\Delta\varphi(z) = 0$. We assume now in addition that boundary points (z,w) are strictly pseudoconvex when $|z|$ is close to 1. Denote by K the compact subset of $\mathbb{D}$ where $\Delta\varphi$ vanishes.

Lemma 4.18. *Consider Ω as above. Then $b\Omega$ contains an analytic disc if and only if K has nonempty (Euclidean) interior.*

Proof. Assume K contains an open disc $B(z_0,r)$. φ is harmonic on $B(z_0,r)$; denote by v a conjugate harmonic function. Then $z \to (z, e^{-(1/2)(\varphi(z)+iv(z))})$ is holomorphic in $B(z_0,r)$, nontrivial, and maps into $b\Omega$. On the other hand, if $\zeta \to (z(\zeta), w(\zeta))$ is a nontrivial analytic disc in $b\Omega$, the equation for $T^{\mathbb{C}}_{(z(\zeta),w(\zeta))}(b\Omega)$ shows that $z'(\zeta) \neq 0$ (note that $w(\zeta) \neq 0$). Thus the projection of (the image of) the disc covers an open set in the z-plane, and $\Delta\varphi = 0$ at all of these points. □

Lemma 4.19. *Consider Ω as above. Then $b\Omega$ satisfies (P_1) if and only if K satisfies (P_1) (equivalently: has empty fine interior).*

Proof. I am indebted to N. Sibony for correspondence on the details of the argument in [272] and on the examples there.

Assume that K satisfies (P_1). To show that $b\Omega$ satisfies (P_1), it suffices to show that the set W of weakly pseudoconvex boundary points satisfies (P_1). This is a consequence of Corollary 4.14, see Remark (i) after the proof of Corollary 4.16. Note that $W = \{(z, e^{-(1/2)\varphi(z)+i\vartheta}) \mid z \in K,\ \vartheta \in [0,2\pi]\}$. On W, the function $-\left(|z|^2 + |w|^2\right)$ equals $-|z|^2 - e^{-\varphi(z)}$; in particular, it is a function of z alone. Because K satisfies (P_1), it can be approximated uniformly on K by functions subharmonic near K. These functions, when viewed as functions of (z, w), are plurisubharmonic near W, and they approximate $-\big(|z|^2 + |w|^2\big)$. By Proposition 4.10 (part (iv)), W satisfies (P_1). Alternatively, one can construct functions λ, $0 \leq \lambda \leq 1$, with large Hessians in the form $\lambda(z, w) = h(z) + A(|w|^2 - e^{-\varphi(z)})^2$, where $h(z)$ has big Hessian near K (and is between zero and one), and A is sufficiently big.

Now assume that W satisfies (P_1). Fix $M > 0$. There exists a plurisubharmonic function λ in a neighborhood of W, $0 \leq \lambda \leq 1$, with its complex Hessian at least M (as a Hermitian form). We may assume that λ is rotationally invariant in w: consider $\tilde{\lambda} := (1/2\pi)\int_0^{2\pi} \lambda(z, we^{i\vartheta})d\vartheta$. Set $h(z) = \lambda(z, e^{-(1/2)\varphi(z)})$. h is defined in a neighborhood of K, and $0 \leq h \leq 1$. The computation of the Hessian of h is simplified by the introduction of the auxiliary function $\mu(z, w) := \lambda(z, e^w)$. It is defined in a neighborhood of $\widetilde{W} = \{(z, w) \mid z \in K,\ \mathrm{Re}(w) = -(1/2)\varphi(z)\}$, and $h(z) = \mu(z, -(1/2)\varphi(z))$. Note that $0 \leq \mu \leq 1$, and the complex Hessian of μ is at least cM for a constant c independent of M. The rotation invariance of λ implies that μ is invariant under translations in the direction of $\mathrm{Im}(w)$. Therefore, $(\partial\mu/\partial w) = (\partial\mu/\partial\overline{w})$. This gives, at points where $z \in K$ (i.e., $\Delta\varphi(z) = 0$)

$$\begin{aligned}\frac{\partial^2 h}{\partial z\partial\bar{z}} &= \frac{\partial}{\partial z}\left(\frac{\partial\mu}{\partial\bar{z}} - \frac{\partial\mu}{\partial\overline{w}}\frac{\partial\varphi}{\partial\bar{z}}\right)\\ &= \frac{\partial^2\mu}{\partial z\partial\bar{z}} - \frac{\partial^2\mu}{\partial w\partial\bar{z}}\frac{\partial\varphi}{\partial z} - \frac{\partial^2\mu}{\partial z\partial\overline{w}}\frac{\partial\varphi}{\partial\bar{z}} + \frac{\partial^2\mu}{\partial w\partial\overline{w}}\left|\frac{\partial\varphi}{\partial z}\right|^2.\end{aligned} \tag{4.45}$$

The right-hand side of (4.45) equals the complex Hessian of μ applied to the vector $(1, (\partial\varphi/\partial\bar{z})$, and so it is at least $cM(1 + |\partial\varphi/\partial\bar{z}|^2) \geq cM$. Thus $\partial^2 h/\partial z\partial\bar{z} > cM$ on K, and therefore in a suitable neighborhood of K. Since c is independent of M and since also $0 \leq h \leq 1$, we are done. □

The examples we are interested in now arise by arranging for the set K to have various properties. To this end, observe the following: for every compact subset K of $\mathbb{D}$, there exists a smooth pseudoconvex complete Hartogs domain with base $\mathbb{D}$ whose set of weakly pseudoconvex boundary points projects onto K. For simplicity of notation, assume K is a compact subset of the disc of radius $1/2$. Let $g_1 \in C^\infty(\mathbb{D})$ with $g_1(z) > 0$ if $z \in \mathbb{D}\setminus K$, and $g_1(z) = 0$ when $z \in K$, and choose $g_2 \in C^\infty(\mathbb{D})$ so that $\Delta g_2 = g_1$. Let $\chi \in C_0^\infty(\mathbb{D})$ with $\chi = 1$ in a neighborhood of $\overline{B(0, 1/2)}$. We take φ in the form $\varphi(z) = \chi(z)g_2(z) + g_3(z)$, where $g_3(z)$ is a radially symmetric function which is identically equal to zero on $\overline{B(0, 1/2)}$ and which agrees with $-\log(1 - |z|^2)$ when $|z|$ is close to 1. Then $\Delta g_3 = \partial^2 g_3/\partial^2 r + (1/r)\partial g_3/\partial r$. We can take g_3 to be monotone, convex, and with $\partial^2 g_3/\partial^2 r$ as big as we want on the compact set $\mathrm{supp}(\chi)\setminus B(0, 1/2)$. In particular, we can arrange for the resulting function φ to be subharmonic on $\mathbb{D}$. Then Ω given by $\Omega = \{(z, w) \mid |w|^2 - e^{-\varphi(z)} < 0\}$ is pseudoconvex, complete, and smooth. For the last assertion, note that when $|z|$ is close to 1, $b\Omega$ agrees with the boundary of the unit ball in $\mathbb{C}^2$.

The following two examples are from [272].

Example. Choose K with empty fine interior, but positive (Lebesgue) measure. Then the resulting domain Ω satisfies (P_1) (Lemma 4.19), hence has a compact $\bar{\partial}$-Neumann operator, yet the set of weakly pseudoconvex boundary points has positive (Euclidean surface) measure on $b\Omega$. The set of infinite type points then also has positive measure. (By Catlin's results, the set of weakly pseudoconvex points of finite type has measure zero, in general. In $\mathbb{C}^2$, the situation is considerably simpler, as the various notions of type agree; see [92], Section 4.3.1.) To obtain such a set K, observe that the projection of a finely open subset of $\mathbb{C}$ onto one of the coordinate axes is open. Indeed, recall from the discussion preceding Proposition 4.17 that if z is a fine interior point of a set E, then there are arbitrarily small positive r such that $bB(z,r) \subseteq E$ (one such circle contained in E would suffice). Take $K = K_1 \times K_2 \subseteq \mathbb{R}^2 \approx \mathbb{C}$, where K_1 and K_2 are compact subsets of $\mathbb{R}$, both with positive Lebesgue measure, and K_1 in addition with empty (Euclidean) interior. If K had nonempty fine interior, its projection onto the first coordinate axis, K_1, would have nonempty (Euclidean) interior.

Example. Choose K with empty Euclidean interior, but nonempty fine interior. Then the boundary of Ω contains no analytic disc (Lemma 4.18), but nonetheless, it does not satisfy (P_1) (Proposition 4.17 and Lemma 4.19). Such a set K is obtained as follows. Denote by $\{a_j\}_{j=1}^{\infty}$ a sequence of points in the unit disc $\mathbb{D}$ that is dense in $\mathbb{D}$. Define $g(z) = \sum_{j=1}^{\infty}(1/j^2)\log(|z-a_j|/3)$, where $|z| < 2$ and the sum is in $\mathcal{L}^1(B(0,2))$ or, equivalently, pointwise (note that the partial sums are decreasing, since $|z-a_j|/3 < 1$). As a limit of a decreasing sequence of subharmonic functions, g is subharmonic (since by the $\mathcal{L}^1$-convergence, g is not identically equal to minus infinity). Choose a finite value c that is assumed by g on $\mathbb{D}$ and set $K = \{z \in \overline{\mathbb{D}} \mid g(z) \geq c\}$. Because g is upper semi-continuous, $\{z \in B(0,2) \mid g(z) < c\}$ is open; therefore K is compact. Because the sequence $\{a_j\}$ is dense in $\mathbb{D}$, K has empty Euclidean interior. On the other hand, the set where $g > c$ is finely open, hence so is its intersection with (the finely open set) $\mathbb{D}$. This intersection is contained in K, and the maximum principle implies that it is nonempty (since g is not constant). Thus there is a nonempty finely open subset of K, i.e., K has nonempty fine interior. All properties of subharmonic functions used here may be found for example in [207], Section 2.1 or in[3], Chapter 3. The latter also contains a similar example of a set K as above (see Example 7.9.3; I am indebted to B. Fuglede for this reference). An explicit construction of sets with empty Euclidean interior, but nonempty fine interior, by removing suitable sequences of discs from $\mathbb{D}$, can also be found in [86], Section 4.

4.8 Obstructions to property (P_q) and to compactness

We have already mentioned that the obvious (but not the only, in view of the preceding example) obstructions to property (P_1) are analytic discs in the boundary. More generally, sets that pick up 'plurisubharmonic hull' ([64], [155], [65]) are obstructions. For a compact set $K \subseteq b\Omega$, set $\widehat{K} = \{z \in b\Omega \mid \lambda(z) \leq \max_{\zeta \in K} \lambda(\zeta)$ for all plurisubharmonic functions λ in $C(\overline{\Omega})\}$. By a result in [272] (mentioned, but not proved, earlier), (P_1) implies that every continuous function on the boundary is the boundary value of a plurisubharmonic function ([272], Théorème 2.1). Therefore, if there is $K \subseteq b\Omega$ such that $\widehat{K} \neq K$, then $b\Omega$ cannot satisfy (P_1). When $n \geq 3$, this can happen even if the boundary does not contain analytic discs ([65], Theorem 5). By contrast, in $\mathbb{C}^2$, a set K can pick up plurisubharmonic hull only via an analytic disc in the boundary ([65], Theorem 4; see also [265], Proposition 1). Thus the previous example shows that there can be obstructions to property (P_1) that do not arise as nontrivial plurisubharmonic hulls (in particular not as analytic discs in the boundary).

When $q > 1$, the role of analytic discs is played by q-dimensional complex manifolds, or analytic polydiscs. A q-dimensional analytic polydisc is a holomorphic map F from the q-fold product $\mathbb{D} \times \cdots \times \mathbb{D}$ to $\mathbb{C}^n$ so that at all points $\zeta \in \mathbb{D}^q$, $F'(\zeta)$ has maximal rank q. In addition, we assume that $F \in C^2(\overline{\mathbb{D}^q})$. As usual, we will also use the term analytic polydisc to denote the image of F.

Lemma 4.20. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $1 \leq q \leq n$. If $b\Omega$ contains a q-dimensional analytic polydisc, it does not satisfy property (P_q).*

Proof. In view of Proposition 4.10 (and the definition of $P_q(\overline{\mathbb{D}^q})$), it suffices to show that C^2 functions that are in P_q of a neighborhood of $\overline{\mathbb{D}^q}$ satisfy a maximum principle on $\overline{\mathbb{D}^q}$. This maximum principle is a consequence of the maximum principle for elliptic operators, as follows. Shrinking the polydisc if necessary, we may consider smooth vector fields $X^1, \ldots, X^q$ on $\mathbb{D}^q$, of type $(1,0)$, such that their push-forwards $F_*(X^1), \ldots, F_*(X^q)$ are orthonormal in $\mathbb{C}^n$. This can be achieved by using $(F_*(X), F_*(Y))_{\mathbb{C}^n}$ as an inner product on the tangent vectors on $\mathbb{D}^q$, and then applying Gram–Schmidt near $0 \in \mathbb{D}^q$ (and then shrinking the polydisc: $\widetilde{F}(z) := F(z/a)$, with $a \gg 1$). Now let $\lambda \in P_q(U) \cap C^2(U)$, where U is a neighborhood of $F(\overline{\mathbb{D}^q})$, and set $h(z) = \lambda(F(z))$. Then h satisfies

$$\begin{aligned}\sum_{j,k=1}^{q} \frac{\partial^2 h(z)}{\partial z_j \partial \bar{z}_k} X_j^1(z)\overline{X_k^1(z)} + \cdots + \sum_{j,k=1}^{q} \frac{\partial^2 h(z)}{\partial z_j \partial \bar{z}_k} X_j^q(z)\overline{X_k^q(z)} \\ = H_1(\lambda, F(z))(F_*(X^1), \overline{F_*(X^1)}) + \cdots + H_1(\lambda, F(z))(F_*(X^q), \overline{F_*(X^q)}) \geq 0.\end{aligned} \tag{4.46}$$

The last inequality follows because $\lambda \in P_q(U)$ and $F_*(X^1), \ldots, F_*(X^q)$ are orthonormal. In other words, h satisfies

$$Lh(z) := \sum_{j,k=1}^{q} a_{j,k}(z) \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}(z) \geq 0, \tag{4.47}$$

with $a_{j,k}(z) = \sum_{s=1}^{q} X_j^s(z)\overline{X_k^s(z)} = \overline{a_{k,j}(z)}$. L is uniformly elliptic on $\overline{\mathbb{D}^q}$:

$$\sum_{j,k=1}^{q} a_{j,k}(z) w_j \overline{w_k} = \sum_{j,k,s} X_j^s(z)\overline{X_k^s(z)} w_j \overline{w_k} = \sum_{s=1}^{q} \left| \sum_{j=1}^{q} X_j^s(z) w_j \right|^2 \geq c|w|^2, \tag{4.48}$$

for a positive constant c. The last inequality holds because $\{X^s(z) \mid s = 1, \ldots, q\}$ is a basis for $\mathbb{C}^q$, $z \in \overline{\mathbb{D}^q}$. The maximum principle for second order elliptic operators (see for example [122], Theorem 1, p. 327, [148], Theorem 3.1) gives $h(z) \leq \max_{\zeta \in b\mathbb{D}^q} h(\zeta)$; $z \in \mathbb{D}^q$. This completes the proof of Lemma 4.20. □

To what extent are the above obstructions to property (P) obstructions to compactness of the $\bar{\partial}$-Neumann operator? Here is a simple example (taken from [213], [205]).

Example. Still denote by $\mathbb{D}$ the unit disc, and let f be holomorphic in $\mathbb{D}$ and square-integrable. The domain Ω is $\mathbb{D} \times \mathbb{D}$ with the edge $(b\mathbb{D} \times b\mathbb{D})$ rounded to make Ω smooth (but keeping the rotational symmetries intact). Then

$$\bar{\partial}(\bar{w} f(z)) = f(z) d\bar{w}. \tag{4.49}$$

Also, if $g(z, w)$ is holomorphic and square-integrable in Ω, then

$$(g, \bar{w} f(z)) = \int_\Omega g(z, w) w \overline{f(z)} dV(z, w). \tag{4.50}$$

Integration in w, with z fixed, gives the integral of a holomorphic function (namely $g(z, w)w$) over a disc centered at 0. This integral equals a multiple of the value at $w = 0$, hence in our case equals 0. Therefore, $\bar{w} f(z)$ is orthogonal to holomorphic functions, i.e.,

$$\bar{w} f(z) = \bar{\partial}^* N_1(f(z) d\bar{w}). \tag{4.51}$$

To see that $\bar{\partial}^* N_1$ is not compact, it suffices to observe that

$$\|\bar{w} f(z)\|_{\mathcal{L}^2(\Omega)} \approx \|f\|_{\mathcal{L}^2(\Omega)} \approx \|f(z) d\bar{w}\|_{\mathcal{L}^2_{(0,1)}(\Omega)}.$$

If $\bar{\partial}^* N_1$ were compact, this would imply that the unit ball in $\mathcal{L}^2(\mathbb{D}) \cap \ker(\bar{\partial})$ is compact. Since $\mathcal{L}^2(\mathbb{D}) \cap \ker(\bar{\partial})$ is infinite dimensional, this is a contradiction. So $\bar{\partial}^* N_1$ is not compact; by Proposition 4.2, neither is N_1.

On the other hand, here is an example (from [142]) of a domain with an analytic disc in the boundary whose $\bar{\partial}$-Neumann operator is nonetheless compact. The reason is that the $\mathcal{L}^2$-theory does not detect the disc.

Example. Set $\Omega := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 < 1,\ 0 < |z| < 1\}$. Ω is the unit ball minus the variety $\{z = 0\}$, and so is nonsmooth (but pseudoconvex). The natural isometry $\mathcal{L}^2_{(0,q)}(\Omega) \hookrightarrow \mathcal{L}^2_{(0,q)}(B(0, 1))$ commutes with $\bar{\partial}$, $q = 0, 1, 2$. This can be checked by an argument as in [24], p. 687. $\bar{\partial}^*$ then enjoys similar commutation properties, and consequently, so do $\square$ and its inverse. In particular, N_1 on Ω inherits compactness from N_1 on the unit ball.

When the boundary is smooth enough, an example like the previous one is not possible in $\mathbb{C}^2$. This observation is usually ascribed to Catlin (unpublished); a proof for the case of a Lipschitz boundary is published in [142]. However, when $n \geq 3$, whether or not an analytic disc in the boundary (say, of a smooth domain) is necessarily an obstruction to compactness is open. The answer is known to be affirmative when the domain is locally convexifiable (Theorem 4.26 below; on these domains, compactness is completely understood). For general domains, only a partial result is known: an analytic disc is known to be an obstruction when it contains a point at which the boundary is strictly pseudoconvex in the $(n-2)$ directions transverse to the disc. In fact, we have the following result from [263] (and [265] for $q = 1$).

Theorem 4.21. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. Let $P \in b\Omega$ and assume that there is an m-dimensional complex manifold $M \subset b\Omega$ through P ($m \geq 1$), and that $b\Omega$ is strictly pseudoconvex at P in the directions transverse to M (this condition is void when $n = 2$). Then the $\bar{\partial}$-Neumann operator N_q on Ω is not compact for $1 \leq q \leq m$.*

Remarks. (i) In particular, if the Levi form is known to have at most one degenerate eigenvalue at each point (the eigenvalue zero has multiplicity at most 1), a disk in the boundary is an obstruction to compactness of N_1. A special case of this is implicit in [182] for domains fibered over a Reinhardt domain in $\mathbb{C}^2$.

(ii) The assumption that the boundary is strictly pseudoconvex at P in the directions transverse to M is somewhat curious. One would expect a flatter boundary to be even more conducive to noncompactness. It turns out that the present methods require some control over the boundary geometry at P; the setup in Theorem 4.21 provides this in a geometrically simple way. (Compare also Theorem 4.26 and its proof below.)

The proof of Theorem 4.21 requires two auxiliary results; both are from [265]. They are of independent interest. The first result gives control over the boundary geometry near a point of a complex submanifold of the boundary.

Lemma 4.22. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, M a complex manifold of dimension m in $b\Omega$, and $P \in M$. Then there is a ball B centered at P, a biholomorphic map $G\colon B \to G(B)$, such that*

(i) $G(P) = 0$,

(ii) $G(M \cap B) = \{w \in G(B) \mid w_{m+1} = \cdots = w_n = 0\}$,

(iii) *the real normal to $G(b\Omega \cap B)$ at points of $G(M \cap B)$ is given by the* $\mathrm{Re}(w_n)$*-axis.*

Proof. We can change coordinates near P so that M is (locally) given by $\{z \mid z_{m+1} = \cdots = z_n = 0\}$ and $P = 0$. When ρ is a defining function for Ω (say with $\partial\rho/\partial z_n \neq 0$ near P), then there exists a real-valued C^∞ function h, near 0, such that the complex normal to $b\Omega$ given by $e^h(\partial\rho/\partial\bar{z}_1, \ldots, \partial\rho/\partial\bar{z}_n)$ has conjugate holomorphic components on M. When $m = 1$ (i.e., M is a disc), this is the conclusion of Lemma 1 in [18]. These methods can be adapted to cover the case of $m > 1$. Alternatively, we can use the ideas that will be developed in Chapter 5; this approach comes from [288].

Specifically, (5.85) and (5.92) show that there is a C^∞ function h, defined on M near 0, that satisfies

$$d_M h(\overline{L}) = \alpha|_M(\overline{L}) = -\frac{1}{|\partial\rho|^2}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\frac{\partial\rho}{\partial\bar{z}_j}\overline{w_k}, \tag{4.52}$$

where $L = \sum_{k=1}^{n} w_k(\partial/\partial z_k)$ is tangential to M (i.e., $w_{m+1} = \cdots = w_n = 0$), and α is the form introduced in Chapter 5. Because α is real, h can now be extended into a full neighborhood of $P = 0$ as a real valued function, so that (4.52) holds near 0 at points of M for any L (not necessarily tangent to M). Now set $L_{n,h} = e^h\sum_{j=1}^{n}(\partial\rho/\partial\bar{z}_j)(\partial/\partial z_j)$. Then (4.52) means that the inner product of $\overline{L}(\overline{L_{n,h}})$ (where $\overline{L}$ acts componentwise) with the conjugate of the complex normal is zero on M:

$$\begin{aligned}\sum_{j=1}^{n}\overline{L}\left(e^h\frac{\partial\rho}{\partial z_j}\right)\frac{\partial\rho}{\partial\bar{z}_j} &= \sum_{j,k=1}^{n}\overline{w_k}\frac{\partial}{\partial\bar{z}_k}\left(e^h\frac{\partial\rho}{\partial z_j}\right)\frac{\partial\rho}{\partial\bar{z}_j}\\ &= e^h\Big((\overline{L}h)|\partial\rho|^2 + \sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\frac{\partial\rho}{\partial\bar{z}_j}\overline{w_k}\Big) = 0.\end{aligned} \tag{4.53}$$

The last equality is from (4.52). The inner products with the conjugates of complex tangential fields vanish by virtue of pseudoconvexity of $b\Omega$ (and regardless of what h is): if $\sum_{j=1}^{n}\zeta_j(\partial/\partial z_j)$ is such a field then (4.53) becomes

$$\sum_{j=1}^{n}\overline{L}\left(e^h\frac{\partial\rho}{\partial z_j}\right)\zeta_j = e^h\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\zeta_j\overline{w_k} = 0. \tag{4.54}$$

We have used that $\sum_{j=1}^{n}(\partial\rho/\partial z_j)\zeta_j = 0$. The last equality holds because L is a Levi null direction, and the Levi form is positive semidefinite. So at any point of M (near 0), $\overline{L}(\overline{L_{n,h}})$ has inner product zero with the vectors in a basis of $\mathbb{C}^n$, hence vanishes. Since L was an arbitrary field tangential to M, this shows that $\overline{L_{n,h}}$ has coefficients that are holomorphic on M.

The holomorphic change of coordinates

$$z \to \left(z_1,\ldots,z_{n-1},\sum_{j=m+1}^{n} z_j e^{h(z_1,\ldots,z_m,0,\ldots,0)}\frac{\partial\rho}{\partial z_j}(z_1,\ldots,z_m,0,\ldots,0)\right) \tag{4.55}$$

takes the complex tangent space of $b\Omega$ (near 0) into the complex hypersurface $z_n = 0$ (we also use z to denote the new coordinates); this is easily computed. In other words, the complex normal is constant at points of (the image of) M, and the real unit normal is of the form $(0,\ldots,0,e^{i\sigma})$. On M, the function σ is pluriharmonic ([105], Section 5; [17], Proposition 3.1; [18], Lemma 1; [14], proof of the main theorem).

Denote by σ_1 a pluriharmonic conjugate (near 0). The final coordinate change

$$(z_1,\ldots,z_n) \to \left(z_1,\ldots,z_{n-1},z_n e^{\sigma_1(z_1,\ldots,z_m,0,\ldots,0)-i\sigma(z_1,\ldots,z_m,0,\ldots,0)}\right) \tag{4.56}$$

rotates the real normal: it becomes constant along M (near 0). Combining the above coordinate changes gives the required biholomorphic map G, and the proof of Lemma 4.22 is complete. □

Remark. The fact that the complex normal is conjugate holomorphic (rather than holomorphic) implies (and is explained by) the following: the bundle given by the complex tangent space $T^{\mathbb{C}}(b\Omega)$ is holomorphic on M. Indeed, writing the condition for membership in $T^{\mathbb{C}}_P(b\Omega)$ at a point P in terms of the complex inner product with the normal at P results in a linear equation whose coefficients vary holomorphically in P on M. Locally, this gives a basis for $T^{\mathbb{C}}(b\Omega)$ that varies holomorphically on M.

The second result concerns the 'source' of noncompactness. Denote by $A(\Omega)$ the Bergman space of Ω, i.e., the subspace of $\mathcal{L}^2(\Omega)$ consisting of holomorphic functions.

Lemma 4.23. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, smooth near the strictly pseudoconvex boundary point P. Assume the pseudoconvex domain $\Omega_1$is contained in Ω, shares the boundary point P with Ω, and is smooth near P. Then the restriction map from $A(\Omega)$ to $A(\Omega_1)$ is not compact.*

Proof. We produce a bounded sequence of functions in $A(\Omega)$ without a subsequence that converges in $A(\Omega_1)$. Choose a sequence $\{P_j\}_{j=1}^{\infty}$ of points on the common interior normal to $b\Omega$ and $b\Omega_1$ at P so that $P_j \to P$. Denote by $K_U(\cdot,\cdot)$ the Bergman kernel function of a domain U (see [259], [207], [176], [81], [243] for properties of this kernel function). For $z \in \Omega$, $K(z,\cdot)$ is the reproducing kernel for $A(\Omega)$ at the point z. It follows from the mean value property of holomorphic functions that $K(z,\cdot)$ is given by the Bergman projection of a function $\varphi \in C_0^{\infty}(\Omega)$, radially symmetric with respect to z, and with $\int_\Omega \varphi = 1$. Therefore, by (4) in Theorem 2.9 and Theorem 3.6, $K(z,\cdot)$ is smooth up to the boundary near P. (More can be said, compare [179], [26], [40].) For $j \geq 1$, define $f_j(z) = K_\Omega(z, P_j)/K_\Omega(P_j, P_j)^{1/2}$. Then $\|f_j\|_{\mathcal{L}^2(\Omega)} = 1$; furthermore, $f_j(z) \to 0$ for all $z \in \Omega$. The last assertion holds because $K(z,\cdot) \in C^{\infty}(\overline{\Omega})$ near P and $K_\Omega(P_j, P_j) \to \infty$. For the last statement, see e.g. [170], Theorem 3.5.1, where the precise asymptotics are determined as follows: $\lim_{j\to\infty} |P_j - P|^{n+1} K_\Omega(P_j, P_j) = (n!/4\pi^n) \times$ (product of the eigenvalues of the Levi form at P); compare also [104]. The proof applies methods discussed in Chapters 1 and 2. This result has been generalized considerably in [53] (compare also [111]).

By the reproducing property of K_{Ω_1}, applied to the function $K_\Omega(\cdot, P_j)$ (viewed as an element of $A(\Omega_1)$), we have

$$\begin{aligned} K_\Omega(P_j, P_j)^2 &= \left(\int_{\Omega_1} K_{\Omega_1}(P_j, w) K_\Omega(w, P_j) dV(w) \right)^2 \\ &\leq \|K_{\Omega_1}(\cdot, P_j)\|^2_{\mathcal{L}^2(\Omega_1)} \|K_\Omega(\cdot, P_j)\|^2_{\mathcal{L}^2(\Omega_1)} \\ &= K_{\Omega_1}(P_j, P_j) \|K_\Omega(\cdot, P_j)\|^2_{\mathcal{L}^2(\Omega_1)}. \end{aligned} \tag{4.57}$$

(This argument is from [141], p. 637.) Therefore,

$$\|f_j\|^2_{\mathcal{L}^2(\Omega_1)} = \frac{\|K_\Omega(\cdot, P_j)\|^2_{\mathcal{L}^2(\Omega_1)}}{K_\Omega(P_j, P_j)} \geq \frac{K_\Omega(P_j, P_j)}{K_{\Omega_1}(P_j, P_j)} \geq C > 0. \tag{4.58}$$

For the second inequality in (4.58), note that Ω_1 is also strictly pseudoconvex at P, so that by Hörmander's result mentioned above, both kernels have the same asymptotics at P: the quotient in (4.58) converges to the (strictly positive) quotient of the determinants of the Levi forms at P of Ω and Ω_1, respectively.

Because $f_j(z) \to 0$ for $z \in \Omega$ (in particular, for $z \in \Omega_1$), (4.58) implies that no subsequence of $\{f_j\}$ converges in $\mathcal{L}^2(\Omega_1)$: by (4.58), $\mathcal{L}^2$ convergence would result in a nonzero limit, contradicting the pointwise convergence to zero (since $\mathcal{L}^2$ convergence of holomorphic functions implies their pointwise convergence, for example via the mean value property). This concludes the proof of Lemma 4.23. □

Proof of Theorem 4.21. The argument, from [265], [263], follows closely [141], Section 4. In turn, that section relies heavily on ideas from [65] and [116]. We assume that $1 \leq q \leq m$, and that N_q is compact. This will lead to a contradiction. Keeping the notation from the previous two lemmas, we denote $G(B \cap \Omega)$ by $\widetilde{\Omega}$. Shrinking B if necessary, we may assume that G is biholomorphic in a neighborhood of $\overline{B \cap \Omega}$. Choose positive numbers $r_2 < r_1 < r_0$ and a ball $\widetilde{B}$ in $\mathbb{C}^{n-m}$ small enough such that $M_0 \times \widetilde{B} \subset \widetilde{\Omega}$, where $M_j = \{w = (w_1, \dots, w_q, 0, \dots, 0) \in M \mid |w| < r_j\}$, $j = 0, 1, 2$. This is possible because the real (unit) normal to the boundary is constant on M (near P). Denote by S_0 the $(n-m)$-dimensional slice of $\widetilde{\Omega}$ through 0 and perpendicular to M. (Further shrinking B if necessary to ensure that this slice is a domain, i.e., is connected.) Because the biholomorphism G preserves the rank of the Levi form at P, S_0 is strictly pseudoconvex at 0. By Lemma 4.23, the restriction map from $A(S_0)$ to $A(\widetilde{B})$ is not compact. Thus, there is a bounded sequence $\{f_j\}_{j=1}^\infty$ none of whose subsequences converges in $A(\widetilde{B})$. The Ohsawa–Takegoshi extension theorem (Theorem 2.17) provides a bounded sequence $\{F_j\}_{j=1}^\infty$ in $A(\widetilde{\Omega})$ such that $F_j = f_j$ on S_0. The forms $\alpha_j := F_j d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_q$ are $\overline{\partial}$-closed on $\widetilde{\Omega}$. By Proposition 4.4, part (2), the $\overline{\partial}$-Neumann operator on $(0, q)$-forms on $B \cap \Omega$ is compact (in view of our assumption that N_q on Ω is compact). Therefore (Proposition 4.2) there is a compact solution operator for $\overline{\partial}$ on $(0, q)$-forms. This gives, via the pullback under G^{-1}, a corresponding compact solution operator on $\widetilde{\Omega}$. This solution operator, when applied to $\{\alpha_j\}_{j=1}^\infty$, produces a (bounded) sequence $\{\beta_j\}_{j=1}^\infty$ in $\mathcal{L}^2_{(0,q-1)}(\widetilde{\Omega})$ with $\overline{\partial}\beta_j = \alpha_j$, $j \in \mathbb{N}$. Note that the last equality still holds when we redefine β_j by omitting all the terms that contain a factor $d\overline{w}_k$ with $k > q$ (since α_j does not contain terms with such a factor); we use $\widehat{\beta}_j$ to denote these new forms. Moreover, $\{\beta_j\}_{j=1}^\infty$ has a convergent subsequence, hence so does $\{\widehat{\beta}_j\}_{j=1}^\infty$ (omitting terms decreases norms). Choose a smooth cutoff function $\chi\colon \mathbb{C}^q \to [0, 1]$ with $\chi(w_1, \dots, w_q) = 1$ when $|(w_1, \dots, w_q)| < r_2$, and $\chi(w_1, \dots, w_q) = 0$ when $|(w_1, \dots, w_q)| > r_1$. Set $\gamma(w) = \chi(w_1, \dots, w_q, 0, \dots, 0) d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_q$. The mean value property for holomorphic functions gives

$$\begin{aligned}|f_j(0,\dots,0,w_{m+1},\dots,w_n) - f_k(0,\dots,0,w_{m+1},\dots,w_n)|\\ = C\left|\int_{|(w_1,\dots,w_m)|<r_0}\langle\alpha_j-\alpha_k,\gamma\rangle dV(w_1,\dots,w_n)\right|\\ = C\left|\int_{|(w_1,\dots,w_m)|<r_0}\langle\widehat{\beta}_j-\widehat{\beta}_k,\vartheta\gamma\rangle dV(w_1,\dots,w_m)\right|\end{aligned} \tag{4.59}$$

(recall that ϑ denotes the formal adjoint of $\bar{\partial}$). Applying the Cauchy–Schwarz inequality and integrating in $(w_{m+1},\dots,w_n)$ gives

$$\|f_j - f_k\|^2_{S_0} \le C\|\widehat{\beta}_j - \widehat{\beta}_k\|^2_{S_0}. \tag{4.60}$$

$\{\widehat{\beta}_j\}_{j=1}^{\infty}$ contains a convergent subsequence. In view of (4.60), this contradicts the fact that $\{f_j\}_{j=1}^{\infty}$ does not. The proof of Theorem 4.21 is complete. □

Theorem 4.21 implies that if N_1 is compact, the set of weakly pseudoconvex boundary points cannot have interior ([265]). Note that this set is in general considerably larger than the set of Levi flat points (where all eigenvalues of the Levi form vanish).

Corollary 4.24. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \ge 2$. If the $\bar{\partial}$-Neumann operator N_1 is compact, then the set of weakly pseudoconvex boundary points has empty interior in $b\Omega$.*

Proof. Let U be a nonempty open set in $b\Omega$ contained in the set of weakly pseudoconvex boundary points. Let $P \in U$ be a point where the Levi form has maximal rank, say m, among the points of U (such a point exists because the rank of the Levi form assumes only finitely many values). Note that $m \le n-2$ (otherwise, the point is a strictly pseudoconvex point). Near P, the rank has to be at least m, hence is equal to m (in particular, it is constant). Therefore, $b\Omega$ is foliated, near P, by complex submanifolds of dimension $n-1-m$ (see for example [133]). At P, $b\Omega$ is strictly pseudoconvex in the directions transverse to the leaf through P (because the rank of the Levi form at P is m). By Theorem 4.21, N_q is not compact for $1 \le q \le n-1-m$. In particular, N_1 is not compact, contradicting the assumption. □

Remark. Both Theorem 4.21 and Corollary 4.24 can be formulated in terms of a compact solution operator for $\bar{\partial}$ (on $(0,q)$-forms and $(0,1)$-forms, respectively). The modifications in the proofs are minor and are left to the reader.

We have already seen that property (P_1) can fail without discs in the boundary (see the discussion following the proof of Lemma 4.19). The examples arise via compact sets in the plane with empty (Euclidean) interior, but nonempty fine interior. Such sets also give rise to examples of domains in $\mathbb{C}^2$ without discs in the boundary, whose $\bar{\partial}$-Neumann operator is not compact. This was realized in [217]. The treatment we give here follows [142]. A Hartogs domain (say in $\mathbb{C}^2$) is called complete if with each point (z,w), it also contains the points $(z,\lambda w)$ for all λ with $|\lambda| \le 1$.

Theorem 4.25. *Assume that K is a compact subset of $\mathbb{C}$ with nonempty fine interior. Then there exists a smooth bounded pseudoconvex complete Hartogs domain $\Omega \subseteq \mathbb{C}^2$ whose weakly pseudoconvex boundary points project onto K and whose $\bar\partial$-Neumann operator N_1 is not compact.*

Remarks. (i) If we choose K in Theorem 4.25 with empty Euclidean interior (in addition to having nonempty fine interior), then $b\Omega$ will contain no analytic disc (Lemma 4.18). Note that in this case, $\mathbb{C} \setminus K$ has infinitely many connected components. Namely, if z is a fine interior point of K, then there are arbitrarily small positive r such that the circle of radius r, centered at z, is contained in K ([166], Theorem 10.14; [3], Theorem 7.3.9). If z is not an (Euclidean) interior point of K, we obtain a sequence of radii tending to 0 so that each annulus between successive circles contains at least one component of $\mathbb{C} \setminus K$ (since the circles are contained in K).

(ii) Although the domains in Theorem 4.25 have noncompact $\bar\partial$-Neumann operator, this operator is globally regular (continuous in Sobolev spaces): a smooth bounded complete Hartogs domain in $\mathbb{C}^2$ without discs in the boundary is 'nowhere wormlike', in the terminology of [49], and consequently has globally regular $\bar\partial$-Neumann operator ([49], Theorem 1).

Proof of Theorem 4.25. Without loss of generality, we may assume that $K \subset \mathbb{D}$. We are going to build a Hartogs domain Ω over $2\mathbb{D}$, the disc of radius 2. It will be given by $|w|^2 < e^{-\varphi(z)}$, where $\varphi \in C^\infty(2\mathbb{D})$. Near $b2\mathbb{D}$, φ will be given by $-\log(4-|z|^2)$; then Ω will be smooth near $\{w = 0\}$ also (compare our discussion following the proof of Lemma 4.19: $b\Omega$ will then agree with the sphere of radius 2 near $\{w = 0\}$). Because we want Ω to be pseudoconvex, we require $\Delta\varphi \geq 0$ (see (4.44)); furthermore (also by (4.44)), we need that K is the zero set of $\Delta\varphi$ (so that the weakly pseudoconvex points project onto K). In addition, we want φ so that the resulting Hartogs domain has noncompact $\bar\partial$-Neumann operator. This will be achieved through a careful construction of φ ([217]).

Let ψ be a nonnegative smooth function on $\mathbb{C}$ that is strictly positive on $\mathbb{C} \setminus K$, and that vanishes to infinite order on K. We treat the case where $\mathbb{D} \setminus K$ has infinitely many connected components (by Remark (i) above, this is the main case of interest); when $\mathbb{D} \setminus K$ has only finitely many components, the proof works with the obvious modifications. Denote by D_j, $j = 1, \dots,$ the connected components of $\mathbb{D} \setminus K$. Set $\psi_j = \psi$ on $\overline{D_j}$ and 0 on $\mathbb{D} \setminus \overline{D_j}$. Then $\psi_j \in C^\infty(\overline{\mathbb{D}})$, and $\chi = \sum_{j=1}^\infty \psi_j$ on $\overline{\mathbb{D}}$. Next, we pick a sequence $\{c_j\}_{j=1}^\infty \subset (0, 1]$ and a strictly increasing sequence of positive integers $\{n_j\}_{j=1}^\infty$ such that: (i) $n_j c_j \int_{D_j} \psi_j = 2\pi$, (ii) $c_{j+1}\|\psi_{j+1}\|_{\mathcal{L}^\infty} \leq c_j\|\psi_j\|_{\mathcal{L}^\infty}$, (iii) $n_j c_{j+1}\|\psi_{j+1}\|_{\mathcal{L}^\infty} \leq 1$, and (iv) n_{j+1} is divisible by n_j. This can be done inductively. First, choose n_1 sufficiently large so that $c_1 := 2\pi/(n_1 \int_{D_1} \psi_1) \leq 1$. Assume $c_1, \dots, c_j$ and $n_1, \dots, n_j$ have been chosen. Choose an integer m sufficiently big so that $c_{j+1} := 2\pi/(mn_j \int_{D_{j+1}} \psi_{j+1}) \leq \min\{1, (c_j\|\psi_j\|_{\mathcal{L}^\infty})/\|\psi_{j+1}\|_{\mathcal{L}^\infty}, 1/(n_j\|\psi_{j+1}\|_{\mathcal{L}^\infty})\}$. Then set $n_{j+1} := mn_j$. Now we set $\tilde\psi := \sum_{j=1}^\infty c_j\psi_j$. The supports of the ψ_j's are disjoint, except for points of K, where all derivatives of ψ, hence of the ψ_j's, vanish. As a consequence, the sum converges in $C^\infty(\overline{\mathbb{D}})$. In particular, $\tilde{\psi} \in C^\infty(\overline{\mathbb{D}})$. Finally, set

$$\varphi(z) := \frac{1}{2\pi}\int_{\mathbb{D}} \log(|z-\zeta|)\tilde{\psi}(\zeta)dA(\zeta). \tag{4.61}$$

Then $\varphi \in C^\infty(\mathbb{D})$, and $\Delta\varphi = \tilde{\psi}$ on $\mathbb{D}$ ($(1/2\pi)\log|z|$ is the fundamental solution for Δ, [122], [299]). As in the discussion following the proof of Lemma 4.19, we can extend φ (from a disc of radius slightly smaller than 1) to a C^∞ function on $2\mathbb{D}$ (still denoted by φ) so that on $2\mathbb{D}\setminus K$, φ is strictly subharmonic and so that near $\{|z|=2\}$, it agrees with $-\log(4-|z|^2)$. We define Ω by $\Omega = \{(z,w) \in \mathbb{C}^2 \mid z \in 2\mathbb{D}, |w|^2 - e^{-\varphi(z)} < 0\}$.

Claim: The $\bar{\partial}$-Neumann operator N_1 on Ω is not compact.

Proof of claim: Recall from the discussion around Proposition 4.17 that $\lambda_1(U)$ denotes the first eigenvalue of the Dirichlet problem for the Laplacian on U, in the classical sense when U is Euclidean open ([77]), but also when U is only finely open (with the appropriate definition of the Dirichlet problem, [135]). No boundary point of D_j is a fine interior point of K. This follows from Theorem 10.14 in [166] (or Theorem 7.3.9 in [3]), which implies that if z is a fine interior point of K, then there are arbitrarily small positive r such that the circle $bB(z,r)$ is contained in K. These circles would 'disconnect' D_j. Therefore, $\mathrm{int}_f K \subseteq W_k := \mathbb{D}\setminus(\bigcup_{j=1}^{k}\overline{D}_j)$, and $\lambda_1(W_k) \le \lambda_1(\mathrm{int}_f K) < \infty$, by the monotonicity of λ_1 ([135]), and by our assumption that $\mathrm{int}_f K \neq \emptyset$. It follows that there exists a sequence of functions $\{v_k\}_{k=1}^{\infty}$, $v_k \in C_0^\infty(W_k)$, $\|v_k\|_{\mathcal{L}^2(\mathbb{D})} = 1$, and $\|\nabla v_k\|_{\mathcal{L}^2(\mathbb{D})} \lesssim 1$. Define Ψ_k by

$$\Psi_k(z) = \frac{1}{2\pi}\int_{\mathbb{D}} \log(|z-\zeta|)\sum_{j=1}^{k} c_j\psi_j(\zeta)\, dA(\zeta). \tag{4.62}$$

Then $\Delta\Psi_k = \sum_{j=1}^{k} c_j\psi_j = 0$ on W_k. Thus Ψ_k is harmonic on the finitely connected domain W_k. Moreover, the properties (i) and (iv) of the sequences $\{c_j\}$ and $\{n_j\}$ imply that the periods of $n_k\Psi_k$, given by (sums of) the integrals $n_k\int_{D_j}\Delta\Psi_k = n_k c_j\int_{D_j}\psi_j$, $1 \le j \le k$, are integer multiples of 2π. Consequently, $n_k\Psi_k$ has a harmonic conjugate Θ_k on W_k whose values are determined up to an integer multiple of 2π. Therefore, $\exp(i\Theta_k)$ is single valued and smooth, and $\exp(n_k\psi_k + i\Theta_k)$ is holomorphic on W_k.

With the help of these functions, we define the following sequence of forms in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ on Ω:

$$u_k = \sqrt{n_k}\, v_k \exp(n_k\varphi + i\Theta_k)w^{n_k-1}\left(\overline{w}d\overline{z} - 2\exp(-2\varphi)\frac{\partial\varphi}{\partial z}d\overline{w}\right). \tag{4.63}$$

Note that the form $\left(\overline{w}d\overline{z} - 2\exp(-2\varphi)\frac{\partial\varphi}{\partial z}d\overline{w}\right) \in \mathrm{dom}(\bar{\partial}^*)$ (see (4.43)), hence so is u_k. In addition, $\|u_k\|_{\mathcal{L}^2_{(0,1)}(\Omega)} \approx 1$ (meaning that these norms are bounded and bounded away from 0), and $u_k \perp u_j$ if $k \neq j$ (different powers of w are orthogonal on discs in the w-variable centered at 0). As a result, $u_k \to 0$ weakly in $\mathcal{L}^2_{(0,1)}(\Omega)$ and hence strongly in $W^{-1}_{(0,1)}(\Omega)$. Therefore, in order to prove the claim, it suffices to show that $\|\bar\partial u_k\|^2 + \|\bar\partial^* u_k\|^2 \lesssim 1$: if this holds, the compactness estimate

$$\|u_k\|^2 \le \varepsilon(\|\bar\partial u_k\|^2 + \|\bar\partial^* u_k\|^2) + C_\varepsilon \|u_k\|^2_{-1} \tag{4.64}$$

cannot hold for all $\varepsilon > 0$.

The Kohn–Morrey formula (Proposition 2.4 with $a \equiv 1$, $\varphi \equiv 0$) gives

$$\|\bar\partial u_k\|^2 + \|\bar\partial^* u_k\|^2 = \int_{b\Omega} \sum_{j,l=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \bar z_l}(u_k)_j \overline{(u_k)_l} + \int_\Omega \sum_{j,l=1}^{2} \left| \frac{\partial (u_k)_l}{\partial \bar z_j} \right|^2, \tag{4.65}$$

where we have temporarily denoted z by z_1 and w by z_2; ρ is a (normalized) defining function. Take $\rho = |w|^2 - \exp(\varphi(z))$, suitably normalized. If f_k denotes the function $\sqrt{n_k}\, v_k \exp(n_k\varphi + i\Theta_k) w^{n_k - 1}$, the right-hand side of (4.65) is dominated independently of k by

$$\begin{aligned} &\int_{b\Omega} \frac{\partial^2 \varphi}{\partial z \partial \bar z} |f_k|^2 + \int_\Omega \left(|f_k|^2 + \left| \frac{\partial f_k}{\partial \bar z} \right|^2 \right) \\ &\quad \lesssim \int_{W_k} \left(n_k |v_k|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar z} + |v_k|^2 + \left| \frac{\partial v_k}{\partial \bar z} \right|^2 + \left| v_k \frac{\partial (n_k \varphi + i\Theta_k)}{\partial \bar z} \right|^2 \right). \end{aligned} \tag{4.66}$$

Note that although Θ_k is multiple valued, derivatives are well defined and single valued. The first term on the right-hand side of (4.66) is dominated by $n_k \max_{\zeta \in W_k} |\partial^2\varphi/\partial z \partial \bar z| \int_{W_k} |v_k|^2 \lesssim n_k \|\tilde\psi\|_{\mathcal{L}^\infty(W_k)} \int_{W_k} |v_k|^2$. By (ii) and (iii), $n_k \|\tilde\psi\|_{\mathcal{L}^\infty(W_k)} \lesssim 1$, and by our choice of v_k, $\int_{W_k} |v_k|^2 \lesssim 1$ as well, so that the first term on the right-hand side of (4.66) is bounded independently of k. Again by our choice of v_k, the second and third terms are bounded independently of k. To control the last term, note that $\partial(n_k \Psi_k + i\Theta_k)/\partial \bar z = 0$, by our choice of Θ_k as conjugate harmonic to Ψ_k. Therefore, we may replace $n_k\varphi + i\Theta_k$ by $n_k(\varphi - \Psi_k)$ in the last term in (4.66). From the definitions (4.61) and (4.62), we obtain via differentiating under the integral and invoking (ii) and (iii)

$$|\varphi(z) - \Psi_k(z)| \lesssim \frac{n_k}{4\pi} \int_{\mathbb{D}} \frac{\sum_{j=k+1}^\infty c_j \varphi_j(\zeta)}{|\zeta - z|} dA(\zeta) \lesssim n_k c_{k+1} \|\varphi_{k+1}\|_{\mathcal{L}^\infty} \lesssim 1. \tag{4.67}$$

Since the $\|v_k\|_{\mathcal{L}^2(W_k)} \lesssim 1$, (4.67) shows that the last term in (4.66) can be bounded (independently of k). This completes the proof of the claim and therefore the proof of Theorem 4.25. □

Remark. The proof of Theorem 4.25 was facilitated by the freedom to construct the domain (i.e., φ) suitably. What if the domain is given? Through a considerably more subtle analysis, Christ and Fu have recently shown that on a smooth bounded pseudoconvex Hartogs domain in $\mathbb{C}^2$, compactness of N_1 and (P_1) are actually equivalent ([86], Theorem 1.1). Their work uses an interesting connection between compactness of N_1 and (P_1) and the behavior of the lowest eigenvalue of certain sequences of magnetic Schrödinger operators and their nonmagnetic counterparts, respectively (in the complete case). These Schrödinger operators live on the base of the domain (an open set in $\mathbb{C}$), and the sequences are indexed by the Fourier variable that arises from the rotational symmetry of the domain (essentially the exponent of w in (4.63)). This connection was established in [143]; a portion of it is implicit in [217]. The main issue then is to show that if the lowest eigenvalues of the operators without magnetic field stay bounded as the Fourier variable tends to infinity, then the eigenvalues of the operators with magnetic field have finite limes inferior (Theorem 1.5 in [86]). This requires a considerably more refined analysis than that in the above proof (see Section 8 in [86]). For further information on Schrödinger operators in this context, the reader may consult [165]. The connection between $\bar\partial$ and Schrödinger operators predates [143]; for more on this connection, see [34], [161], [162].

4.9 Locally convexifiable domains

There is a class of domains where the analysis, the geometry, and the potential theory mesh perfectly. This is the class of locally convexifiable domains. We say that the domain Ω is locally convexifiable if for every boundary point there is a neighborhood, and a biholomorphic map defined on this neighborhood, that takes the intersection of the domain with the neighborhood onto a convex domain. Pseudoconvexity is a local property of the boundary; consequently, locally convexifiable domains are pseudoconvex. Note that convex domains are Lipschitz (see for example [218], Section 1.1.8); therefore, so are locally convexifiable domains. The following theorem comes from [141], [142]. By an affine variety of dimension q we mean a (relatively) open subset of a q-dimensional affine linear subspace of $\mathbb{C}^n$.

Theorem 4.26. *Let Ω be a bounded domain in $\mathbb{C}^n$ which is locally convexifiable. Then the following are equivalent:*

(i) *The $\bar\partial$-Neumann operator N_q is compact.*

(ii) *$b\Omega$ does not contain any analytic variety of dimension $\geq q$.*

(iii) *$b\Omega$ satisfies (P_q).*

If Ω is convex, then (i)–(iii) *are also equivalent to*

(iv) *$b\Omega$ does not contain an affine variety of dimension $\geq q$.*

Remarks. (i) The equivalence of (ii) and (iv) on convex domains is a manifestation of the general principle that for convex domains, questions of orders of contact may be decided by considering affine (in particular; smooth) varieties ([219], [49], [304]). This is in stark contrast to the situation on general pseudoconvex boundaries (see [89], [92]).

(ii) Complex manifolds in general (but smooth) pseudoconvex boundaries are studied in [18].

(iii) On convex domains, compactness of N_1 has been shown in [181] to be equivalent to a condition called property-K in [181]. This property encodes information concerning the holomorphic structure of subdomains having the property that the restriction operator from the Bergman space of the domain to the Bergman space of the subdomain is not compact. This latter property plays a crucial role in establishing failure of compactness on certain domains, see the role Lemmas 4.23 and 4.27 play in the proofs of Theorems 4.21 and 4.26, respectively. Interestingly, the definition of property-K involves the Kobayashi metric of the domain.

(iv) Given Lemmas 4.23, 4.27, and the remarks in (iii), the following question arises. Determine necessary and sufficient conditions on a pair of domains $\Omega_1 \subseteq \Omega_2$, with $b\Omega_1 \cap b\Omega_2 \neq \emptyset$, for compactness of the restriction operator $A(\Omega_2)$ to $A(\Omega_1)$. In this context, see [181], [182].

(v) It is in general not easy to decide whether a given (nonconvex) domain is locally convexifiable; in particular, there is no known characterization in terms of local boundary data. For some results on domains in $\mathbb{C}^2$, we refer the reader to [202], [203]. For the issue of local versus global convexifiability, see [237].

The analogue of Lemma 4.23 required in the proof of Theorem 4.26 is as follows. It is implicit in [141], and is made explicit in [182].

Lemma 4.27. *Let Ω be a bounded convex domain in $\mathbb{C}^n$ with $0 \in b\Omega$. Let $\Omega_{1/2} = \{(1/2)z \mid z \in \Omega\}$. Then the restriction from $A(\Omega)$ to $A(\Omega_{1/2})$ is not compact.*

Proof. The strategy is the same as in the proof of Lemma 4.23, but the tactics need to be adjusted. Choose a point $P \in \Omega$ on the normal to a supporting hyperplane at $0 \in b\Omega$, and a sequence of points P_j on the line segment from 0 to P such that $\lim_{j\to\infty} P_j = 0$. Note that convexity of Ω implies the following geometric arrangement. There exists $r > 0$ and j_0 such that for $j \geq j_0$, the translate of $\Omega \cap B(0, r)$ by the vector $\overrightarrow{0P_j}$ is contained in Ω. As in the proof of Lemma 4.23, we set $f_j(z) = K_\Omega(z, P_j)/\sqrt{K_\Omega(P_j, P_j)}$, where $K_\Omega(\cdot, \cdot)$ denotes the Bergman kernel function of the domain Ω (see again [259], [207], [176], [81]). Then $\|f_j\|_{\mathcal{L}^2(\Omega)} = 1$.

We also need that $K(P_j, P_j) \to \infty$ when $j \to \infty$. This is true on general pseudoconvex domains that satisfy a mild geometric condition (a so called outer cone condition) that is satisfied by convex domains, see e.g. [176], Theorem 6.1.17. The point is to show that there exists a function h in $A(\Omega)$ that tends to infinity along the sequence $\{P_j\}_{j=1}^{\infty}$. Then $|h(P_j)| = |\langle h, K_\Omega(\cdot, P_j)\rangle| \leq \|h\|_{\mathcal{L}^2(\Omega)}\|K_\Omega(\cdot, P_j)\|_{\mathcal{L}^2(\Omega)} = \|h\|_{\mathcal{L}^2(\Omega)}\sqrt{K_\Omega(P_j, P_j)} \to \infty$. In our situation (but not on general pseudoconvex domains) this can be done easily. Without loss of generality, we may assume that the normal to the supporting hyperplane at 0 is the x_n-axis, and that $\Omega \subseteq \{x_n > 0\}$. Then the function $h(z) = \log(z_n)$ will do, where log denotes the principal branch.

Functions in $A(\Omega)$ can be approximated in norm of $A(\Omega)$ by functions in $A(\Omega) \cap C^\infty(\overline{\Omega})$: it suffices to consider dilates with respect to a point in Ω. For $h \in A(\Omega) \cap$

$C^\infty(\overline{\Omega})$, $h(P_j)/K_\Omega(P_j,P_j) \to 0$ as $j \to \infty$ (since $K_\Omega(P_j,P_j) \to \infty$ as $j \to \infty$). In other words, $A(\Omega) \ni h \to h(P_j)/\sqrt{K_\Omega(P_j,P_j)}$ gives a sequence of continuous linear functionals on $A(\Omega)$ that converge weakly to zero on a dense subset of $A(\Omega)$. Moreover, the norms of these functionals are bounded by 1:

$$|h(P_j)| = |\langle h, K_\Omega(P_j,\cdot)\rangle| \le \|h\|_{\mathcal{L}^2(\Omega)}\|K_\Omega(P_j,\cdot)\|_{\mathcal{L}^2(\Omega)} = \|h\|_{\mathcal{L}^2(\Omega)}\sqrt{K(P_j,P_j)}.$$

Consequently, the sequence of functionals converges to zero weakly on all of $A(\Omega)$. In particular, for $z \in \Omega$ fixed, $h = K_\Omega(\cdot,z) \in A(\Omega)$, so that

$$\overline{f_j(z)} = K_\Omega(P_j,z)/\sqrt{K_\Omega(P_j,P_j)} \to 0 \quad \text{as } j \to \infty.$$

(For this argument, compare [239] and [176], Section 7.6.)

We also have

$$\|f_j\|^2_{\mathcal{L}^2(\Omega_{1/2})} \ge \frac{K_\Omega(P_j,P_j)}{K_{\Omega_{1/2}}(P_j,P_j)} = 2^{-n}\frac{K_\Omega(P_j,P_j)}{K_\Omega(2P_j,2P_j)}. \tag{4.68}$$

The inequality on the left follows as in (4.58), while the equality on the right results from the transformation rule for the Bergman kernel under the map $z \to 2z$ that maps $\Omega_{1/2}$ biholomorphically onto Ω. To obtain a lower bound on the right-hand side of (4.68), we use a localization property of the Bergman kernel function. Namely, there exists a constant C such that for $z \in B(0,r/2)$, $K_\Omega(z,z) \ge CK_{\Omega\cap B(0,r)}(z,z)$ ([176], Theorem 6.3.5). Consequently, when j is big enough,

$$\begin{aligned} K_\Omega(P_j,P_j) &\ge CK_{\Omega\cap B(0,r)}(P_j,P_j) \\ &= CK_{T_j(\Omega\cap B(0,r))}(2P_j,2P_j) \ge CK_\Omega(2P_j,2P_j), \end{aligned} \tag{4.69}$$

where T_j denotes translation by the vector $\overrightarrow{0P_j}$ (under which the kernel is invariant). The last inequality in (4.69) holds because $T_j(\Omega\cap B(0,r)) \subseteq \Omega$. Combining (4.68) and (4.69) shows that the $\|f_j\|_{\mathcal{L}^2(\Omega_{1/2})}$ stay bounded away from 0. Together with the pointwise convergence to 0, this implies that $\{f_j\}_{j=1}^\infty$, which is bounded in $A(\Omega)$, has no subsequence that converges in $A(\Omega_{1/2})$. This completes the proof of Lemma 4.27. □

Proof of Theorem 4.26. We follow [141], [142]. The key to the equivalence of (ii) and (iv) (only (iv) $\Rightarrow$ (ii) is nontrivial) on a convex domain is the observation that if V is a q-dimensional variety in $\mathbb{C}^n$, then its convex hull $\widehat{V}$ contains an affine variety of dimension q. (See e.g. [82] for information about complex varieties.) This can be seen by induction on the dimension n. When $n = 1$, this is clear. Let now $V \subseteq \mathbb{C}^n$. If $\widehat{V}$ has nonempty interior (in $\mathbb{C}^n$), we are done. If the interior of $\widehat{V}$ is empty, there do not exist $2n$ line segments with endpoints in V that are linearly independent over $\mathbb{R}$. Therefore V is contained in a real hyperplane. After changing coordinates, we may assume this hyperplane is $\{x_n = 0\}$. The open mapping property of nonconstant holomorphic functions, applied to the coordinate function z_n restricted to the regular part of V, shows that V is actually contained in the complex hyperplane $\{z_n = 0\}$, and the induction is complete.

We now show that not (ii) implies not (iv). So let V be a variety contained in $b\Omega$ of dimension at least q. Without loss of generality, we may assume that $P = 0 \in V$, P is a regular point of V, and that $\{x_n = 0\}$ is a supporting hyperplane for $\overline{\Omega}$ at P. Near P, V is a smooth manifold, say V_P, and the argument from the previous paragraph shows that V_P is contained in $\{z_n = 0\}$. Therefore, its convex hull is contained in both $\overline{\Omega}$ and $\{z_n = 0\}$, hence in $b\Omega$. But by the previous paragraph, this convex hull contains an affine variety of the same dimension as V_P (at least q), that is, (iv) fails.

Next, we come to the equivalence of (i)–(iii) on locally convexifiable domains. We already know that (iii) implies (i): that is the content of Theorem 4.8. It remains to show that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii).

For the proof that (i) $\Rightarrow$ (ii) we argue indirectly and assume that N_q is compact on Ω and there is a q-dimensional variety in contained in $b\Omega$. This will lead to a contradiction. Choose a regular point P in the variety. By assumption, there is a neighborhood U of P, and a biholomorphism F on U that maps $\Omega \cap U$ onto a convex domain. By considering the inverse image of a small ball centered at $F(P)$, we may assume that U is strictly pseudoconvex, and that F is biholomorphic in a neighborhood of $\overline{U}$. By part (2) of Proposition 4.4, the $\bar{\partial}$-Neumann operator $N_q^{\Omega\cap U}$ on $\Omega \cap U$ is compact (because N_q^{Ω} is). By Proposition 4.2, there is a compact solution operator for $\bar{\partial}\colon \mathcal{L}^2_{(0,q-1)}(\Omega \cap U) \to \mathcal{L}^2_{(0,q)}(\Omega \cap U)$. Via the pull backs F^* and $(F^{-1})^*$ (which commute with $\bar{\partial}$), this solution operator gives an analogous compact solution operator on $F(\Omega \cap U)$.

The boundary of $F(\Omega \cap U)$ contains a q-dimensional complex manifold (through $F(P)$). Because of the equivalence of (ii) and (iv) on convex domains that we already proved, $bF(\Omega \cap U)$ contains a q-dimensional affine variety. In suitable coordinates, we have that $M_R := \{(z_1, \dots, z_q, 0, \dots, 0) \mid |z_1|^2 + \cdots + |z_q|^2 \le R^2\} \subset bF(\Omega \cap U)$ for some $R > 0$. Denote by $\Omega_1 \subseteq \mathbb{C}^{n-q}$ the domain $\{(z_{q+1}, \dots, z_n \mid (0, \dots, 0, z_{q+1}, \dots, z_n) \in F(\Omega \cap U)$. Note that Ω_1 is convex. Set $\Omega_{1/2} := (1/2)\Omega_1$. Then $M_{R/2} \times \Omega_{1/2} \subseteq F(\Omega \cap U)$. (This is a special case of a general fact about convex sets, see [261], Theorem 6.1. In our case, each point in this set is the midpoint of a line segment in $M_R \times \Omega_1$ and so belongs to $\overline{F(\Omega \cap U)}$. But $M_{R/2} \times \Omega_{1/2}$ is open, so must then be contained in the interior of $\overline{F(\Omega \cap U)}$, i.e. in $F(\Omega \cap U)$, where equality holds by convexity.) From now on the proof proceeds exactly as the proof of Theorem 4.21. The roles of M_0 and $\widetilde{B}$ are played by $M_{R/2}$ and $\Omega_{1/2}$, respectively. The resulting contradiction shows that if N_q is compact, $b\Omega$ cannot contain a complex variety of dimension $\ge q$.

We now show that (ii) implies (iii). For this, we require the following function algebra result. For a compact set X, denote by $H(X)$ the Banach algebra given by the closure in $C(X)$ of the functions holomorphic in a neighborhood of X. A closed subset E of X is a peak set for $H(X)$ if there exists $f \in H(X)$ such that $f(z) = 1$ on E and $|f(z)| < 1$ on $X \setminus E$. For more information on uniform algebras, we refer the reader to [144], [280]. Let X be compact and convex, $1 \le q \le n$. Let $P \in X$

and assume X contains no variety of dimension $\geq q$ through P. Then there exists a complex affine subspace L of dimension $\leq (q-1)$ through P such that $X \cap L$ is a peak set for $H(X)$. The simplest case, $q = 1$, simply says that when X contains no variety of positive dimension through P, then P is a peak point for $H(X)$. For a proof of this fact, we refer the reader to [141], Proposition 3.2. Note that $X \cap L \subseteq bX$.

So assume now that $b\Omega$ contains no analytic variety of dimension $\geq q$. We must show that $b\Omega$ satisfies (P_q). By Corollary 4.13, it suffices to do this locally. Fix a boundary point $Q \in b\Omega$. Using a biholomorphism F that convexifies $b\Omega$ near Q, as in the proof of (i) $\Rightarrow$ (ii), together with the function algebraic result from the previous paragraph (applied to a suitable convex domain on the image side), we obtain the following. There is a neighborhood U of Q so that $\Omega \cap U$ is a biholomorphic image (under F^{-1}) of a convex domain. Furthermore, there is $r > 0$ such that through each point z of $b\Omega \cap \overline{B(Q,r)}$ there is a (connected) complex manifold M_z of dimension $m \leq (q-1)$ through the point so that $M_z \cap (\overline{\Omega \cap U})$ is a peak set for $H(\overline{\Omega \cap U})$. Also, $M_z \cap (\overline{\Omega \cap U}) \subseteq b\Omega$. Moreover, in a neighborhood of $\overline{\Omega \cap U}$, there are holomorphic coordinates $w_j = g_j(\zeta)$, $1 \leq j \leq n$, where $g_1, \dots, g_n$ are holomorphic in a neighborhood of $\overline{\Omega \cap U}$, $g(z) = 0$, and M_z is given by $g_{m+1} = \cdots = g_n = 0$.

Let μ be a q-Jensen measure for z with respect to $P_q(b\Omega \cap \overline{B(Q,r)})$. Because the absolute values of functions in $H(\overline{\Omega \cap U})$ belong to $P_q(b\Omega \cap \overline{B(Q,r)})$, it follows that μ is supported on $M_z \cap (b\Omega \cap \overline{B(Q,r)}) = M_z \cap \overline{B(Q,r)}$. If $m = 0$, we have that μ is supported on $\{z\}$. If $m > 0$, consider the function

$$h(\zeta) = -\sum_{l=1}^{m} |g_l(\zeta)|^2 + C \sum_{l=m+1}^{n} |g_l(\zeta)|^2, \tag{4.70}$$

where $C > 0$ is big enough so that $h \in P_q(\overline{\Omega \cap U}) \hookrightarrow P_q(b\Omega \cap \overline{B(Q,r)})$. To see that this can be done, let $\zeta \in \overline{\Omega \cap U}$, and let $\underline{t}_1, \dots, \underline{t}_q$ be orthonormal in $\mathbb{C}^n$. Then

$$\begin{aligned} &\sum_{s=1}^{q} \sum_{j,k=1}^{n} \frac{\partial^2 h(\zeta)}{\partial z_j \partial \bar{z}_k} (\underline{t}_s)_j \overline{(\underline{t}_s)_k} \\ &\quad = -\sum_{s=1}^{q} \sum_{l=1}^{m} |D_{\underline{t}_s} g_l(\zeta)|^2 + C \sum_{s=1}^{q} \sum_{l=m+1}^{n} |D_{\underline{t}_s} g_l(\zeta)|^2, \end{aligned} \tag{4.71}$$

where $D_{\underline{t}_s} g_l$ denotes the derivative of g_l in the direction of $\underline{t}_s$. It now suffices to observe that compactness gives that the negative term on the right-hand side of (4.71) is bounded for $\zeta \in \overline{\Omega \cap U}$ (and $\underline{t}_1, \dots, \underline{t}_q$ orthonormal), and that $\sum_{s=1}^{q} \sum_{l=m+1}^{n} |D_{\underline{t}_s} g_l(\zeta)|^2$ assumes a (strictly) positive minimum. The last assertion follows because $1 \leq m \leq q-1$ and the g_l have linearly independent gradients: if $\sum_{s=1}^{q} \sum_{l=m+1}^{n} |D_{\underline{t}_s} g_l(\zeta)|^2 = 0$, then $\underline{t}_1, \dots, \underline{t}_q$ and the gradients of g_l at ζ, $m+1 \leq l \leq n$, would be linearly independent. That is, there would be $q + n - m \geq q + n - (q-1) = n+1$ linearly independent vectors in $\mathbb{C}^n$, which is impossible. Therefore, C can be chosen big enough so that the right-hand side of (4.71) is positive; by Lemma 4.7, the sum of any q eigenvalues of h is nonnegative, i.e., $h \in P_q(\overline{\Omega \cap U})$ (Lemma 4.9). Note that on $M_z \setminus \{z\}$, $h < 0$, while $h(z) = 0$. Because we already know that a q-Jensen measure μ for z is supported on $M \cap \overline{B(Q,r)}$, this implies that it is actually supported on $\{z\}$. We have shown: if μ is a q-Jensen measure for z with respect to $P_q(b\Omega \cap \overline{B(Q,r)})$, then it is supported on $\{z\}$, for all $z \in b\Omega \cap \overline{B(Q,r)}$). By Proposition 4.10, $b\Omega \cap \overline{B(Q,r)}$ satisfies (P_q). $Q \in b\Omega$ was arbitrary, so Corollary 4.13 shows that $b\Omega$ satisfies (P_q). This completes the proof of (ii) $\Rightarrow$ (iii), and thus the proof of Theorem 4.26. ☐

Remarks. (i) In the case of a convex domain, the proof of the implication (i) $\Rightarrow$ (ii) above can be streamlined somewhat; and it gives a proof of (i) $\Rightarrow$ (iv) that uses only the existence of a compact solution operator for $\bar{\partial}$ on forms with holomorphic coefficients (see [141] for details). Consequently, on convex domains, the existence of such a compact restricted solution operator is already equivalent to compactness of the $\bar{\partial}$-Neumann operator. For spectral properties of this restricted solution operator, also in the context of weighted spaces, see [163] and their references.

(ii) Pseudoconvex Reinhardt domains are locally convexifiable at most of their points, and the above ideas yield results on this class of domains as well. In [142], the following theorem is proved (Theorem 5.2). Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$, $1 \le q \le n$. If $b\Omega$ does not contain an analytic variety of dimension $\ge q$, then it satisfies (P_q), and consequently, the $\bar{\partial}$-Neumann operator N_q on $(0,q)$-forms is compact. Indeed, the argument above in the proof of (ii) $\Rightarrow$ (iii) shows that $b\Omega$ satisfies (P_q) locally near every point not on a coordinate hyperplane (these are precisely the points where the domain can be locally convexified). Consequently, the part of the boundary where all coordinates are nonzero (this is not a compact set) is a countable union of compact sets satisfying (P_q). A little extra work is needed to see that the rest of the boundary satisfies (P_q); this follows from logarithmic convexity, using tools from Section 8, Part II, of [261] for unbounded convex domains (details are in [142]). The conclusion is that the boundary is a countable union of compact sets satisfying (P_q); by Corollary 4.14, it satisfies (P_q) as well. We remark that a weaker version of the result had been shown earlier in [167]. Note in particular (in view of Lemma 4.20) that the boundary of a bounded pseudoconvex Reinhardt domain therefore satisfies (P_q) if and only if it does not contain q-dimensional analytic varieties. For a characterization when pseudoconvexity is not assumed (but $q = 1$), see [117], Proposition 3.1.

4.10 A variant of property (P_q)

There is a condition potentially weaker than property (P_q) that still implies compactness. It arises from (P_q) by replacing the uniform boundedness condition in the family of functions by that of having uniformly bounded gradient in the metric induced by the Hessian of the functions. This condition was introduced, and shown to imply compactness, a few years ago by McNeal ([221]). We say that Ω satisfies property $(\widetilde{P_q})$ if the following holds: there is a constant C such that for all $M > 0$ there exists a C^2 function λ in a neighborhood U (depending on M) of $b\Omega$ with

$$\text{(i)} \quad \sum_{|K|=q-1}{}' \left|\sum_{j=1}^{n} \frac{\partial\lambda}{\partial z_j}(z) w_{jK}\right|^2 \leq C \sum_{|K|=q-1}{}' \sum_{j,k} \frac{\partial^2\lambda}{\partial z_j \partial\bar{z}_k}(z) w_{jK}\overline{w_{kK}} \tag{4.72}$$

and

$$\text{(ii)} \quad \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\lambda}{\partial z_j \partial\bar{z}_k}(z) w_{jK}\overline{w_{kK}} \geq M|w|^2, \quad w \in \Lambda^{(0,q)}(z), \tag{4.73}$$

for $z \in U$. Note that this implies in particular that $H_q(\lambda, z)$ is positive semidefinite, i.e., $\lambda \in P_q(U)$. The condition of having the gradients uniformly bounded in the metric induced by the Hessians ((4.72)) appears in various contexts in the literature; we refer the reader to the references in [221], see also [223], [36]. Note that the constant C in front of the sum on the right-hand side of (4.72) is immaterial: by scaling $\lambda \to a\lambda$ ($a \in \mathbb{R}$), it can be made arbitrarily small.

Lemma 4.28. *We have* (i) $(P_q) \Rightarrow (\widetilde{P_q})$, *and* (ii) $(\widetilde{P_q}) \Rightarrow (\widetilde{P_{q+1}})$.

Proof. To see (i), denote by λ_M the family of functions from the definition of (P_q). Then $\mu_M := e^{\lambda_M}$ will work for $(\widetilde{P_q})$. This is seen by a simple computation. Now assume that Ω satisfies $(\widetilde{P_q})$, and let λ_M be the family of functions from the definition. Then the same family works for $(\widetilde{P_{q+1}})$. We already know that for (4.73) (in view of Lemma 4.7): if the sum of the smallest q eigenvalues of the complex Hessian of λ_M is at least M, then the sum of the smallest $q+1$ is also at least M (since there can be at most $(q-1)$ negative eigenvalues, the extra eigenvalue that is added is positive). To check (4.72), let $w \in \Lambda^{(0,q+1)}(z)$. Then

$$\sum_{|K|=q}{}' \left|\sum_{j=1}^{n} \frac{\partial\lambda_M}{\partial z_j} w_{jK}\right|^2 = \sum_{(m,S)}{}' \left|\sum_{j=1}^{n} \frac{\partial\lambda_M}{\partial z_j} w_{jmS}\right|^2, \tag{4.74}$$

where (m, S) is an increasing q-tuple. Letting m and S vary independently increases the sum. We may also interchange the indices j and m on the right-hand side of (4.74) (since $w_{mjS} = -w_{jmS}$). Applying (4.72) to $v_m := \sum'_{|K|=q} w_{mK} d\bar{z}_K \in \Lambda^{(0,q)}(z)$, we obtain that the expressions in (4.74) are bounded by

$$\begin{aligned} &\sum_{m=1}^{n} \sum_{|S|=q-1}{}' \left|\sum_{j=1}^{n} \frac{\partial\lambda_M}{\partial z_j} w_{mjS}\right|^2 \\ &\quad \leq \sum_{m=1}^{n} C\left(\sum_{|S|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\lambda_M}{\partial z_j \partial\bar{z}_k} w_{mjS}\overline{w_{mkS}}\right) \\ &\quad = qC \sum_{|K|=q}{}' \sum_{j,k=1}^{n} \frac{\partial^2\lambda_M}{\partial z_j \partial\bar{z}_k} w_{jK}\overline{w_{kK}}. \end{aligned} \tag{4.75}$$

The equality on the last line of (4.75) follows because in the sum in the second line of (4.75), each term $w_{jK}\overline{w_{kK}}$ occurs precisely q times as $w_{jmS}\overline{w_{kmS}} = w_{mjS}\overline{w_{mkS}}$. (Similar arguments have occurred earlier in the proof of Proposition 4.5.) □

Remarks. (i) The proof of (i) in Lemma 4.28 shows that (P_q) is $(\widetilde{P_q})$, but with a uniform bound on the functions λ_M, in addition to (4.72) and (4.73). It is not hard to see that on the level of an individual function, having the complex gradient bounded in the metric induced by the complex Hessian (as in $(\widetilde{P_1})$) does not force the function to be bounded, see the discussion in [221]. However, what happens on the level of families, as in (4.72) and (4.73), is not at all clear. That is, whether $(\widetilde{P_q})$ is strictly more general than (P_q) is not understood. Note that they are equivalent for classes of domains where compactness of N_q is equivalent to (P_q) (since then $(\widetilde{P_q}) \Rightarrow N_q$ compact $\Rightarrow (P_q)$), in particular for locally convexifiable domains (Theorem 4.26) and for smooth pseudoconvex Hartogs domains in $\mathbb{C}^2$ (see the remark following the proof of Theorem 4.25). (P_1) and $(\widetilde{P_1})$ are also equivalent for compact sets in the complex plane ([143], Lemma 7). This means in particular that an analytic disc in the boundary is an obstruction to $(\widetilde{P_1})$ as well (via pulling back the plurisubharmonic functions to the unit disc in the plane; see also [286], Section 2.2).

(ii) $(\widetilde{P_1})$ also occurs naturally when one constructs Stein neighborhood bases of $\overline{\Omega}$ via a suitable power of a defining function which is (strictly) plurisubharmonic outside $\overline{\Omega}$ (whose sublevel sets then give the basis); compare [272], proof of Théorème 4.1, in particular equation (4.4); [273], proof of Theorem 2.4. For a construction of a Stein neighborhood basis from (P_1) when the boundary is only C^1, see [159].

(iii) In [293], Takegoshi introduced a sufficient condition for compactness of the $\bar{\partial}$-Neumann operator that may be viewed as a precursor to property $(\widetilde{P_1})$, in the sense that the condition replaces the boundedness condition on the family of functions in property (P_1) by a certain condition on the gradients. In fact, Takegoshi's condition implies (4.72) and (4.73) (for $q = 1$) when $z \in b\Omega$ is a weakly pseudoconvex point and $w \in T_z^{(1,0)}(b\Omega)$. This already implies the full property $(\widetilde{P_1})$: it suffices to have (4.72) and (4.73) for z weakly pseudoconvex and w in the null space of the Levi form at z ([74]).

We now state and prove McNeal's result ([221]):

Theorem 4.29. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n$. Assume Ω satisfies property $(\widetilde{P_q})$. Then N_q is compact.*

Proof. This proof is from [286]; the strategy is the same as in [221], but the details are different. We only treat the case where Ω is (at least C^2) smooth and refer the reader to the proof of Corollary 2.13 for the regularization procedure (from [283]) in the general case.

In view of Lemma 4.2, it suffices to establish compactness of $\bar{\partial}^* N_q$ and $\bar{\partial}^* N_{q+1}$. Because $(\widetilde{P_q}) \Rightarrow (\widetilde{P_{q+1}})$ (Lemma 4.28), it suffices to establish compactness of $\bar{\partial}^* N_q$ (assuming $(\widetilde{P_q})$). Compactness of $\bar{\partial}^* N_q$ is equivalent to compactness of its adjoint

$(\bar{\partial}^* N_q)^*$, and this is what we will establish. Note that on $\ker(\bar{\partial})^\perp$, $(\bar{\partial}^* N_q)^*$ gives the canonical solution to $\bar{\partial}^*$, while on $\ker(\bar{\partial})$, it is identically 0.

For $M > 0$, denote by λ_M the function from the definition of $(\widetilde{P_q})$. We may assume that λ_M is a smooth function on all of $U_M \cup \Omega$ (replacing U_M by a slightly smaller set if necessary; (4.72) and (4.73) are still only assumed near $b\Omega$); this function is still denoted by λ_M. The starting point is the Kohn–Morrey–Hörmander formula (Proposition 2.4 with $a \equiv 1$). We use λ_M as the weight φ. If $u \in \ker(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*) \subseteq \mathcal{L}^2_{(0,q)}(\Omega)$, we obtain from Proposition 2.4

$$\int_\Omega {\sum_{|K|=q-1}}' \sum_{j,k} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}}\, e^{-\lambda_M} \le \|\bar{\partial}^*_{\lambda_M} u\|^2_{\lambda_M}. \tag{4.76}$$

A comment is in order. In Proposition 2.4, it is assumed that $u \in C^1(\overline{\Omega})$. If u is only in $\ker(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*)$, we use Proposition 2.3 (density of smooth forms in the graph norm): expressing $\bar{\partial}^*_{\lambda_M} u$ in terms of $\bar{\partial}^* u$ and 0-th order terms ((4.77) below) shows that (4.76), augmented by the $\bar{\partial}$ term on the right-hand side, passes from the approximating sequence to (4.76) for $u \in \ker(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*)$. From (2.23) we have

$$\bar{\partial}^*_{\lambda_M} u = \bar{\partial}^* u + {\sum_{|K|=q-1}}' \left(\sum_{j=1}^n \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar{z}_K, \tag{4.77}$$

whence

$$\begin{aligned} e^{-\lambda_M/2} \bar{\partial}^*_{\lambda_M} u &= e^{-\lambda_M/2} \bar{\partial}^* u + e^{-\lambda_M/2} {\sum_{|K|=q-1}}' \left(\sum_{j=1}^n \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar{z}_K \\ &= \bar{\partial}^* \left(e^{-\lambda_M/2} u \right) + \frac{1}{2} e^{-\lambda_M/2} {\sum_{|K|=q-1}}' \left(\sum_{j=1}^n \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar{z}_K. \end{aligned} \tag{4.78}$$

Taking squared $\mathcal{L}^2$-norms and combining with (4.76) gives

$$\begin{aligned} &\int_\Omega {\sum_{|K|=q-1}}' \sum_{j,k} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}}\, e^{-\lambda_M} \\ &\quad \le C_1 \left\| \bar{\partial}^* \left(e^{-\lambda_M/2} u \right) \right\|^2 + C_1 \int_\Omega {\sum_{|K|=q-1}}' \left| \sum_{j=1}^n \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right|^2 e^{-\lambda_M}, \end{aligned} \tag{4.79}$$

with a constant C_1 independent of λ_M (hence of M). Therefore, because λ_M satisfies (4.72), the integrand in the last term in (4.79) can be absorbed into the integrand on the left-hand side for $z \in U_M \cap \Omega$. We use here that we may take the constant C in (4.72) as small as we please; in particular, we may assume that $C_1 C \leq 1/2$. The result is

$$\int_\Omega \sum_{|K|=q-1}' \sum_{j,k} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}}\, e^{-\lambda_M} \leq 2C_1 \left\| \bar{\partial}^* \left(e^{-\lambda_M/2} u\right)\right\|^2 + C_M \|u\|^2_{\mathcal{L}^2_{(0,q)}(\Omega \setminus \overline{U_M})}. \tag{4.80}$$

Now we use (4.73) for $z \in U_M \cap \Omega$ and observe that integrals involving u (but not derivatives of u) over $\Omega \setminus \overline{U_M}$ can be moved to the right-hand side and estimated by $C_M \|u\|^2_{\mathcal{L}^2_{(0,q)}(\Omega\setminus\overline{U_M})}$. This gives the estimate

$$\|e^{-\lambda_M/2} u\|^2 \leq \frac{2C_1}{M} \left\| \bar{\partial}^* \left(e^{-\lambda_M/2} u\right)\right\|^2 + C_M \|u\|^2_{\mathcal{L}^2_{(0,q)}(\Omega\setminus\overline{U_M})}. \tag{4.81}$$

We estimate the W^1-norm of u on $\Omega\setminus\overline{U_M}$ by interior elliptic regularity (see the estimate (2.85)) and then use the interpolation inequality $\|u\|^2_{\mathcal{L}^2(\Omega\setminus\overline{U_M})} \leq \text{s.c.}\|u\|^2_{W^1(\Omega\setminus\overline{U_M})} + \text{l.c.}\|u\|^2_{W^{-1}(\Omega\setminus\overline{U_M})}$. Inserting the result into (4.81) gives, after collecting terms (and when s.c. is chosen small enough)

$$\|e^{-\lambda_M/2} u\|^2 \leq \frac{C_2}{M} \left\| \bar{\partial}^* \left(e^{-\lambda_M/2} u\right)\right\|^2 + C_M \|e^{-\lambda_M/2} u\|^2_{-1}, \tag{4.82}$$

with a constant C_2 that does not depend on M. We have also used that the (-1)-norm on $\Omega\setminus\overline{U_M}$ is dominated by the (-1)-norm on Ω. (4.82) looks like the compactness estimate we need for the canonical solution operator to $\bar{\partial}^*$, except that it is on $e^{-\lambda_M/2}\ker(\bar{\partial})$, rather than on $\ker(\bar{\partial})$ itself. To handle this difficulty, we use the Bergman projection $P_q\colon \mathcal{L}^2_{(0,q)}(\Omega) \to \ker(\bar{\partial})$, and its weighted variant $P_{q,\lambda_M/2}$ (the orthogonal projection with respect to $(\cdot,\cdot)_{\lambda_M/2}$). The point is that every $v \in \ker(\bar{\partial})$ is the projection of a form $e^{-\lambda_M/2} u$ with $u \in \ker(\bar{\partial})$:

$$v = P_q\big(e^{-\lambda_M/2}(P_{q,\lambda_M/2})(e^{\lambda_M/2} v)\big). \tag{4.83}$$

To verify (4.83), it suffices to pair with a $\bar{\partial}$-closed form g:

$$\begin{aligned}\big(e^{-\lambda_M/2} P_{q,\lambda_M/2}(e^{\lambda_M/2} v),\ g\big) &= \big(P_{q,\lambda_M/2}(e^{\lambda_M/2} v),\ g\big)_{\lambda_M/2} \\ &= \big(e^{\lambda_M/2} v,\ g\big)_{\lambda_M/2} = (v,\ g).\end{aligned} \tag{4.84}$$

Because $v \in \ker(\bar{\partial})$, (4.83) follows. Note that $u = (P_{q,\lambda_M/2})(e^{\lambda_M/2} v) \in \ker(\bar{\partial})$, and if $v \in \text{dom}(\bar{\partial}^*)$, then so is u (the domains of $\bar{\partial}^*$ and $\bar{\partial}^*_{\lambda_M/2}$ agree, and are preserved under the respective Bergman projections; compare the discussion around (2.21) to (2.23)). Observing that the Bergman projection is norm-nonincreasing, and that $\bar{\partial}^* g = \bar{\partial}^* P_q g$ for any $g \in \text{dom}(\bar{\partial}^*)$, we obtain from (4.82), for $v \in \ker(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$,

$$\|v\|^2 \lesssim \frac{1}{M} \|\bar{\partial}^* v\|^2 + C_M \|e^{-\lambda_M/2}(P_{q,\lambda_M/2})(e^{\lambda_M/2} v)\|^2_{-1}. \tag{4.85}$$

Because the canonical solution operator to $\bar{\partial}^*$ is continuous in $\mathcal{L}^2$-norms, the norm in the last term of (4.85) is compact not only with respect to $\|v\|$ (that is, $v \to e^{-\lambda_M/2}(P_{q,\lambda_M/2})(e^{\lambda_M/2}v)$ is compact from $\mathcal{L}^2_{(0,q)}(\Omega)$ to $W^{-1}_{(0,q)}(\Omega)$) , but also with respect to $\|\bar{\partial}^* v\|$. Since M was arbitrary, this implies, in view of Lemma 4.3, that $(\bar{\partial}^* N_q)^*$, restricted to $\ker(\bar{\partial})^{\perp}$, is compact. But $(\bar{\partial}^* N_q)^*$ vanishes on $\ker(\bar{\partial})$, so it is compact on $\mathcal{L}^2_{(0,q-1)}(\Omega)$ (to $\mathcal{L}^2_{(0,q)}(\Omega)$). This completes the proof of Theorem 4.29. □

4.11 Geometric sufficient conditions for compactness

To what extent property (P_q) (or $(\widetilde{P_q})$) is necessary for compactness of N_q on general (say sufficiently smooth) domains is open. So far, all the compactness results discussed rely on property (P_q) (or $(\widetilde{P_q})$) (note that this also holds for domains of finite type: subellipticity is established via families of plurisubharmonic functions that have properties stronger than those required for property (P_q), see [70], [283]). We now discuss an approach to compactness that does not proceed via verifying (P_q) (or $(\widetilde{P_q})$. This approach, introduced by the author in [285] for domains in $\mathbb{C}^2$, relies on geometric conditions satisfied by the boundary. More specifically, the condition is formulated in terms of short time flows generated by certain complex tangential vector fields. We use the following notation. If Z is a real vector field defined on some open subset of $b\Omega$ (or of $\mathbb{C}^2$), we denote by $\mathcal{F}^t_Z$ the flow generated by Z. Let $B(P,r)$ denote the open ball of radius r, centered at P. Note that for a smooth domain in $\mathbb{C}^2$ the various notions of type coincide (see e.g. [92]), so that we need not specify in which sense finite type is to be understood.

Theorem 4.30. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$, and denote by K the (compact) set of boundary points of infinite type. Assume that there exist constants $C_1, C_2 > 0$, C_3 with $1 \le C_3 < 3/2$, and a sequence $\{\varepsilon_j > 0\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} \varepsilon_j = 0$ so that the following holds. For every $j \in \mathbb{N}$ and $P \in K$ there is a (real) complex tangential vector field $Z_{P,j}$ of unit length defined in some neighborhood of P in $b\Omega$ with $\max|\operatorname{div} Z_{P,j}| < C_1$ such that $\mathcal{F}^{\varepsilon_j}_{Z_{P,j}}(B(P, C_2(\varepsilon_j)^{C_3}) \cap K) \subseteq b\Omega \setminus K$. Then the $\bar{\partial}$-Neumann operator N_1 on Ω is compact.*

Here is a geometrically simple special case. We say that $b\Omega \setminus K$ satisfies a complex tangential cone condition, if there is a (possibly small) open cone C in $\mathbb{C}^2 \approx \mathbb{R}^4$ having the following property. For each $P \in K$ there exists a complex tangential direction so that when C is moved in a rigid motion to have its vertex at P and axis in that complex tangential direction, then $C \cap b\Omega$ is contained in $b\Omega \setminus K$.

Corollary 4.31. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Assume that $b\Omega \setminus K$ satisfies a complex tangential cone condition. Then the $\bar{\partial}$-Neumann operator N_1 on Ω is compact.*

Proof. The complex tangential cone condition implies that the assumptions in Theorem 4.30 are satisfied with C_3 equal to 1. □

It is noteworthy that the conditions in the theorem are ‘minimal’, in the sense that modulo the size of the lower bound on the radius given by $C_2\varepsilon_j^{C_3}$, they are necessary for compactness. Assume that Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ with compact $\bar{\partial}$-Neumann operator. This implies that $b\Omega$ contains no analytic disc (Theorem 4.21). Now let $P \in b\Omega$, and let Z be a real vector field defined near P, complex tangential to the level surfaces of a defining function for Ω. Then Z and JZ span (over $\mathbb{R}$) the complex tangent space to $b\Omega$ at each point near P. Here J denotes as usual the complex structure map of $\mathbb{C}^n$: $J(z_1, \dots, z_n) = (iz_1, \dots, iz_n)$. For $0 \le \theta \le 2\pi$, denote by Z^θ the vector field $Z^\theta = (\cos\theta)Z + (\sin\theta)JZ$. For $\varepsilon > 0$, set $M_\varepsilon := \{\mathcal{F}^t_{Z^\theta}(P) \mid 0 \le t < \varepsilon, 0 \le \theta \le 2\pi\}$. M_ε is a smooth two (real) dimensional submanifold of $b\Omega$. It was shown in [63] (see also [265], Lemma 3, for a generalization) that if all points of M_ε are weakly pseudoconvex points of $b\Omega$, then M_ε is an analytic disc. Catlin’s argument is as follows. It suffices to see that M_ε has only complex tangents, that is, the tangent space to M_ε is invariant under J ([6], Proposition 1.3). So we need to show that every tangent vector to M_ε is a (real) linear combination of Z and JZ. Set $f(t,\theta) = \mathcal{F}^t_{Z_\theta}(P)$; then $\frac{\partial f}{\partial t} = Z^\theta = (\cos\theta)Z + (\sin\theta)JZ$. It remains to consider $\frac{\partial f}{\partial \theta}$. Regard θ as a parameter in the initial value problem that determines $\mathcal{F}^t_{Z_\theta}(P)$. Then $\partial f/\partial\theta$ satisfies the initial value problem

$$\begin{gathered} \left(\frac{\partial f}{\partial \theta}\right)' = \left(\frac{\partial Z^\theta}{\partial x}\right)\frac{\partial f}{\partial \theta} - (\sin\theta)Z + (\cos\theta)JZ, \\ \frac{\partial f}{\partial \theta}(0,\theta) = 0, \end{gathered} \tag{4.86}$$

Here, the last two terms on the right-hand side of the differential equation come from $\partial Z^\theta/\partial\theta$, x temporarily denotes a coordinate on $b\Omega$ near P, and $\partial Z^\theta/\partial x$ is the real Jacobian matrix (the derivative). Because the initial value problem (4.86) has a unique solution, it suffices to see that it has a solution in the form $a_\theta(t)Z + b_\theta(t)JZ$ to conclude that $\frac{\partial f}{\partial \theta}$ is a linear combination of Z and JZ at each point of M_ε. Substituting this ansatz into (4.86) and taking into account that $(\partial Z/\partial x)JZ - (\partial JZ/\partial x)Z = [JZ, Z]$ gives the following conditions for $(a_\theta(t), b_\theta(t))$:

$$\begin{gathered} a'_\theta Z + b'_\theta JZ + (b_\theta\cos\theta - a_\theta\sin\theta)\,[Z, JZ] = -(\sin\theta)Z + (\cos\theta)JZ, \\ a_\theta(0) = b_\theta(0) = 0. \end{gathered} \tag{4.87}$$

If all points of M_ε are weakly pseudoconvex boundary points of $b\Omega$, then the commutator $[Z, JZ]$ is complex tangential and so is a linear combination of Z and JZ (with coefficients that are smooth). (4.87) then becomes a linear inhomogeneous initial value problem for $(a_\theta(t), b_\theta(t))$ that has a (unique) solution. So the initial value problem (4.86) indeed has a solution in the form $aZ + bJZ$, and we are done. We have shown: if $b\Omega$ contains no analytic disc, then for all $\varepsilon > 0$, M_ε contains strictly pseudoconvex boundary points. If $\mathcal{F}^{\varepsilon'}_{Z^{\theta_0}}(P)$ is such a point, then there is $r > 0$ such that $\mathcal{F}^{\varepsilon'}_{Z^{\theta_0}}(B(P,r)) \subseteq b\Omega \setminus K$. Since P was arbitrary, this shows that the vector fields required in Theorem 4.30, without the lower bound on r, exist. Moreover, since there are domains without discs in their boundaries whose $\bar{\partial}$-Neumann operator fails to be compact (Theorem 4.25 and Remark (i) following the theorem), we also obtain that with no lower bound on the radius, the conclusion in Theorem 4.30 does not hold. While our bound is most likely rather far from optimal, it is not clear what an optimal bound would be, nor even whether such a bound exists, that is, whether compactness (in $\mathbb{C}^2$) can be characterized in these terms.

Remarks. (i) Whether or not domains that satisfy the assumptions in Theorem 4.30 satisfy property (P) is open. That is , whether Theorem 4.30 can furnish domains with compact $\bar{\partial}$-Neumann operator, but without (P), remains to be seen.

(ii) Theorem 4.30 and Corollary 4.31 have been generalized to $\mathbb{C}^n$ in [231]. There it no longer suffices that the vector fields $Z_{P,j}$ are complex tangential: it is important that they are in directions where the Levi form is no more than a constant (independent of P and ε) times its minimal eigenvalue (at the point). Remark (i) also applies.

Proof of Theorem 4.30. We follow [285]. To show that N_1 is compact, we will establish a compactness estimate: for all $\varepsilon > 0$, there is a constant C_ε such that

$$\|u\|^2 \leq \varepsilon(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) + C_\varepsilon\|u\|_{-1}^2 \quad \text{for all } u \in C^\infty_{(0,1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*). \tag{4.88}$$

Note that because Ω is C^∞-smooth, (4.88) then holds for $u \in \mathcal{L}^2_{(0,1)}(\Omega)$ (by density in the graph norm, Proposition 2.3).

The idea of the proof is simple. To estimate the $\mathcal{L}^2$-norm of u near a point P of K, we write u there in terms of the values in a patch that meets $b\Omega$ inside the set of finite type points, plus the integral of $Z_{P,j}u$ along a short integral curve of $Z_{P,j}$. The contribution from the patch is easily handled by subelliptic estimates. The contribution form the integral is dominated by the square root of the length of the curve (i.e., $\sqrt{\varepsilon_j}$) times the $\mathcal{L}^2$-norm of $Z_{P,j}u$. Because $Z_{P,j}$ is complex tangential, the latter is estimated by $\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2$ on domains in $\mathbb{C}^2$. Making this argument precise raises overlap issues as well as divergence issues for the $Z_{P,j}$. These are handled by the uniformity built into the assumptions in Theorem 4.30.

The details are as follows. The fields $Z_{P,j}$ can be extended from $b\Omega$ inside Ω by a fixed distance by letting them be constant along the real normal; we still denote these fields by $Z_{P,j}$. Note that these fields are complex tangential to the level sets of the boundary distance, and we still have the bounds on $|\mathrm{div}\, Z_{P,j}|$.

Fix $j \in \mathbb{N}$. Because of the overlap issues that will arise, we cover K somewhat more carefully than usual by balls resulting from the assumptions in the theorem. Namely, the family of closed balls $\{\overline{B(P, \frac{C_2}{10}(\varepsilon_j)^{C_3}} \mid P \in K\}$ covers K. This implies that there exists a subfamily of pairwise disjoint balls so that the corresponding balls of radius $\frac{C_2}{2}(\varepsilon)^{C_3}$ (i.e., the radius has increased five fold), hence the open balls of radius $C_2\varepsilon^{C_3}$, still cover K (see for example [306], Theorem 1.3.1). Because K is compact, we thus obtain a finite subfamily of balls $\{B(P_k, C_2\varepsilon^{C_3}) \mid 1 \leq k \leq N\}$ that covers K so that the closed balls $\overline{B(P_k, \frac{C_2}{10}(\varepsilon)^{C_3})}$ are pairwise disjoint. To simplify notation, we denote the vector fields $Z_{P_k,j}$ by Z_k (recall that j is fixed). We may assume, by decreasing C_2 in the theorem if necessary, that $\mathcal{F}^{\varepsilon_j}_{Z_k}\left(B\left(P_k, C_2(\varepsilon)^{C_3}\right) \cap K\right)$ is relatively compact in $b\Omega \setminus K$. Therefore there are open subsets U_k, $1 \leq k \leq N$, of $\overline{\Omega}$, such that

$$K \cap B\left(P_k, C_2(\varepsilon_j)^{C_3}\right) \subseteq U_k \subseteq B\left(P_k, C_2(\varepsilon_j)^{C_3}\right) \tag{4.89}$$

and

$$\mathcal{F}^{\varepsilon_j}_{Z_k}(U_k \cap b\Omega) \cap K = \emptyset. \tag{4.90}$$

It is understood here that the U_k's are chosen in a sufficiently small neighborhood of $b\Omega$ so that the flows $\mathcal{F}^t_{Z_k}$, $1 \leq k \leq N$, are defined for $0 \leq t \leq \varepsilon_j$ for all initial points in U_k.

For $u \in C^\infty_{(0,1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar\partial^*)$, we have

$$\|u\|_0^2 = \int\limits_{\left(\bigcup_{k=1}^N U_k\right)\cap\Omega} |u|^2 + \int\limits_{\Omega\setminus\left(\bigcup_{k=1}^N U_k\right)} |u|^2. \tag{4.91}$$

$\Omega \setminus \bigcup_{k=1}^N U_k$ meets the boundary in a closed subset of $b\Omega \setminus K$, and we can apply subelliptic pseudo-local estimates (see (3.75) and the discussion there) to estimate the second term on the right-hand side of (4.91): there is an open subset V of Ω that contains $\Omega \setminus \bigcup_{k=1}^N U_k$, an $s > 0$, and a constant C such that the restriction of u to V belongs to $W^s(V)$ and satisfies the estimate

$$\|u\|^2_{W^s_{(0,1)}(V)} \leq C\left(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2\right). \tag{4.92}$$

Interpolation of Sobolev norms gives

$$\begin{aligned}\int\limits_{\Omega\setminus\left(\bigcup_{k=1}^N U_k\right)} |u|^2 \leq \|u\|_{\mathcal{L}^2_{(0,1)}(V)} &\leq \frac{\varepsilon_j}{C}\|u\|^2_{W^s_{(0,1)}(V)} + C_j\|u\|^2_{W^{-1}_{(0,1)}(V)} \\ &\leq \varepsilon_j\left(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2\right) + C_j\|u\|^2_{-1}.\end{aligned} \tag{4.93}$$

We now come to the first term on the right-hand side of (4.91). For $1 \leq k \leq N$ fixed, we have

$$\begin{aligned}\int_{U_k\cap\Omega} |u|^2 &= \int\limits_{U_k\cap\Omega} \left|u(\mathcal{F}^{\varepsilon_j}_{Z_k}(x)) - \int\limits_0^{\varepsilon_j} Z_k u\left(\mathcal{F}^t_{Z_k}(x)\right)dt\right|^2 dV(x) \\ &\leq 2\int\limits_{U_k\cap\Omega} \left|u\left(\mathcal{F}^{\varepsilon_j}_{Z_k}(x)\right)\right|^2 dV(x) \\ &\quad + 2\int\limits_{U_k\cap\Omega} \left|\int\limits_0^{\varepsilon_j} Z_k u\left(\mathcal{F}^t_{Z_k}(x)\right)dt\right|^2 dV(x).\end{aligned} \tag{4.94}$$

To estimate the first term on the right-hand side of (4.94), we use a change of coordinates under the diffeomorphism $\mathcal{F}_{Z_k}^{-\varepsilon_j} : \mathcal{F}_{Z_k}^{\varepsilon_j}(U_k \cap \Omega) \to U_k \cap \Omega$. We denote the (positive) real Jacobian determinant of this diffeomorphism by $\det(\partial x/\partial y)$. Then

$$\begin{aligned}\int_{U_k\cap\Omega} \left|u\left(\mathcal{F}_{Z_k}^{\varepsilon_j}(x)\right)\right|^2 dV(x) &= \int_{\mathcal{F}_{Z_k}^{\varepsilon_j}(U_k\cap\Omega)} |u(y)|^2 \det(\partial x/\partial y)dV(y) \\ &\leq C_k \int_{\mathcal{F}_{Z_k}^{\varepsilon_j}(U_k\cap\Omega)} |u(y)|^2 dV(y).\end{aligned} \tag{4.95}$$

The last term in (4.95) can be estimated using subelliptic estimates (in view of (4.90)), and an argument analogous to the one that led to (4.93) gives

$$\int_{\mathcal{F}_{Z_k}^{\varepsilon_j}(U_k\cap\Omega)} |u(y)|^2 dV(y) \leq \frac{\varepsilon_j}{C_k N}(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2) + C_j\|u\|_{-1}^2. \tag{4.96}$$

The Cauchy–Schwarz inequality and Fubini's theorem give for the second term on the right-hand side of (4.94)

$$\begin{aligned}&\int_{U_k\cap\Omega} \left|\int_0^{\varepsilon_j} Z_k u(\mathcal{F}_{Z_k}^t(x))dt\right|^2 dV(x) \\ &\quad\leq \varepsilon_j \int_{U_k\cap\Omega}\int_0^{\varepsilon_j} \left|Z_k u\left(\mathcal{F}_{Z_k}^t(x)\right)\right|^2 dt\, dV(x) \\ &\quad= \varepsilon_j \int_0^{\varepsilon_j}\int_{U_k\cap\Omega} \left|Z_k u\left(\mathcal{F}_{Z_k}^t(x)\right)\right|^2 dV(x)\, dt \\ &\quad= \varepsilon_j \int_0^{\varepsilon_j}\int_{\mathcal{F}_{Z_k}^t(U_k\cap\Omega)} |Z_k u(y)|^2 \det(\partial x/\partial y)dV(y)\, dt \\ &\quad\leq 2\varepsilon_j \int_0^{\varepsilon_j}\int_{\mathcal{F}_{Z_k}^t(U_k\cap\Omega)} |Z_k u(y)|^2\, dV(y)\, dt.\end{aligned} \tag{4.97}$$

In the last estimate we have used the bound $|\operatorname{div} Z_k| \leq C_1$. Because $\operatorname{div} Z_k$ measures the rate of change of the volume under the flow generated by Z_k (compare for example [294], Section 2 of Chapter 2), this implies that $\det(\partial x/\partial y) \leq \exp(tC_1) \leq \exp(\varepsilon_j C_1) \leq 2$ for j big enough (i.e., ε_j small enough; it suffices to establish the compactness estimate (4.88) for small ε).

Putting together estimates (4.94)–(4.97) and adding over k, we estimate the first term on the right-hand side of (4.91):

$$\begin{aligned}\int_{\left(\bigcup_{k=1}^{N} U_k\right)\cap\Omega} |u|^2 &\le \sum_{k=1}^{N} \int_{U_k\cap\Omega} |u|^2 \\ &\le \sum_{k=1}^{N}\Bigg[\frac{2\varepsilon_j}{N}\left(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2\right) + C_j\|u\|_{-1}^2 \\ &\qquad + 4\varepsilon_j \int_0^{\varepsilon_j}\int_{\mathcal{F}_{Z_k}^t(U_k\cap\Omega)} |Z_k u(y)|^2\, dV(y)\, dt\Bigg].\end{aligned} \tag{4.98}$$

That is,

$$\begin{aligned}\int_{\left(\bigcup_{k=1}^{N} U_k\right)\cap\Omega} |u|^2 &\le 2\varepsilon_j\left(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2\right) + C_j\|u\|_{-1}^2 \\ &\quad + 4\varepsilon_j\int_0^{\varepsilon_j}\left(\sum_{k=1}^{N}\int_{\mathcal{F}_{Z_k}^t(U_k\cap\Omega)} \left|Z_k u(y)\right|^2 dV(y)\right)dt.\end{aligned} \tag{4.99}$$

C_j denotes as usual a constant whose value depends only on j, but whose actual value may change at each occurrence.

We now address the overlap of the sets $\mathcal{F}_{Z_k}^t(U_k\cap\Omega)$, $1 \le k \le N$, in the sum in the last term in (4.99). If $Q \in \mathcal{F}_{Z_k}^t(U_k\cap\Omega)\cap\mathcal{F}_{Z_m}^t(U_m\cap\Omega)$, then by the triangle inequality, the distance between $\mathcal{F}_{Z_k}^{-t}(Q)$ and $\mathcal{F}_{Z_m}^{-t}(Q)$ is no more than $2t \le 2\varepsilon_j$ (recall that $|Z_k| = 1$). Consequently, $B\left(P_m, \frac{C_2}{10}(\varepsilon_j)^{C_3}\right) \subseteq B\left(P_k, 2\varepsilon_j + 2C_2(\varepsilon_j)^{C_3} + \frac{C_2}{10}(\varepsilon_j)^{C_3}\right)$. Since the balls $B\left(P_l, \frac{C_2}{10}(\varepsilon_j)^{C_3}\right)$, $1 \le l \le N$, are pairwise disjoint, comparison of volumes gives that no more than $C(C_2)(\varepsilon_j)^{4-4C_3}$ of them can be contained in $B\left(P_k, 2\varepsilon_j + 2C_2(\varepsilon_j)^{C_3} + \frac{C_2}{10}(\varepsilon_j)^{C_3}\right)$, where $C(C_2) = \left[(2+2C_2+C_2/10)/(C_2/10)\right]^4$. It follows that for t fixed, no point of Ω is contained in more than $C(C_2)(\varepsilon_j)^{4-4C_3}$ of the sets $\mathcal{F}_{Z_k}^t(U_k\cap\Omega)$.

If ρ is a defining function for Ω, denote by L the complex tangential field $(\partial\rho/\partial z_2)\partial/\partial z_1 - (\partial\rho/\partial z_1)\partial/\partial z_2$, normalized near $b\Omega$ so that $|L| \equiv 1$ in a tubular neighborhood of $b\Omega$ that contains $\bigcup_{k=1}^{N} U_k$ (shrink the U_k if necessary). Then $Z_k = a_k L + \overline{a_k L}$ with $|a_k| = 1$, $1 \le k \le N$, and $|Z_k u|^2 \le 2\left(|Lu|^2 + |\overline{L}u|^2\right)$.

(4.99) and the conclusion reached at the end of the last paragraph then give

$$\begin{aligned}
&\int_{\left(\cup_{k=1}^{N} U_k\right)\cap\Omega} |u|^2 \\
&\quad\le 2\varepsilon_j\left(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2\right) + C_j\|u\|_{-1}^2 \\
&\qquad + 4\varepsilon_j \int_0^{\varepsilon_j} 2C(C_2)(\varepsilon_j)^{4-4C_3}\, 2\int_\Omega \left(|Lu(y)|^2 + |\bar{L}u(y)|^2\right) dV(y)\, dt \\
&= 2\varepsilon_j\left(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2\right) + C_j\|u\|_{-1}^2 \\
&\qquad + 8C(C_2)\varepsilon_j^{6-4C_3}\, 2\int_\Omega \left(|Lu(y)|^2 + |\bar{L}u(y)|^2\right) dV(y).
\end{aligned} \tag{4.100}$$

In $\mathbb{C}^2$, the integral in the last term in (4.100) is dominated by $\|\bar{\partial}u\|^2 + |\bar{\partial}^* u\|^2$ (Lemma 4.32 below). Combining (4.93) and (4.100) then shows that there is a constant independent of j such that for j sufficiently big

$$\|u\|^2 \le C\left(\varepsilon_j + \varepsilon_j^{6-4C_3}\right)\left(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2\right) + C_j\|u\|_{-1}^2. \tag{4.101}$$

By assumption, $6-4C_3 > 0$, so that $\varepsilon_j + \varepsilon_j^{6-4C_3} \to 0$ as $j \to \infty$. Therefore, (4.101) gives the desired compactness estimate, and the proof of Theorem 4.30 is complete. □

The following lemma is a special case of Théorème 3.1 in [101] (and the proof is from there). Derridj shows that for domains in $\mathbb{C}^n$, an estimate like (4.102), where norms of complex tangential derivatives are controlled by $\|\bar{\partial}u\| + \|\bar{\partial}^* u\|$ (a so-called 'maximal estimate') holds if and only if all the eigenvalues of the Levi form are comparable. In $\mathbb{C}^2$, this condition is trivially satisfied. Further interesting properties of these domains are in [102], [185], [186].

Lemma 4.32. *Let $\Omega = \{\rho(z) < 0\}$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$, $L = \sum_{j=1}^{2} a_j(z)(\partial/\partial z_j)$ a type $(1,0)$ vector field with coefficients in $C^\infty(\bar{\Omega})$ which near $b\Omega$ is complex tangential to the level sets of ρ. Then there is a constant C such that*

$$\|Lu\|^2 + \|\bar{L}u\|^2 \le C\left(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2\right), \quad u \in \operatorname{dom}(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*). \tag{4.102}$$

Proof. Near $b\Omega$, L is a smooth multiple of the field $(\partial\rho/\partial z_2)(\partial/\partial z_1) - (\partial\rho/\partial z_1)(\partial/\partial z_2)$. We may assume that L is normalized; a smooth factor only affects the constant in (4.102). Furthermore, it suffices to prove (4.102) for $u \in C^\infty_{(0,1)}(\bar{\Omega})$, via approximation of u in the graph norm. In view of the Kohn–Morrey formula (Proposition 2.4 with

$a \equiv 0$, $\varphi \equiv 0$), the estimate for $\|\overline{L}u\|_0^2$ is clear. Integration by parts gives for $\|Lu\|_0^2$

$$\begin{aligned}\int_\Omega Lu\,\overline{Lu} &= -\int_\Omega \overline{L}Lu\,\bar{u} + O(\|Lu\|\|u\|)\\ &= -\int_\Omega [\overline{L}, L]u\,\bar{u} - \int_\Omega L\overline{L}u\,\bar{u} + O(\|Lu\|\|u\|)\\ &= \int_\Omega [L, \overline{L}]u\,\bar{u} + \int_\Omega \left|\overline{L}u\right|^2 + O(\|Lu\|\|u\| + \|\overline{L}u\|\|u\|).\end{aligned} \tag{4.103}$$

The last term in (4.103) can be estimated by s.c.$\|Lu\|^2 +$ l.c.$\big(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2\big)$ (since $\|u\|^2$ is also dominated by $\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2$; see (2.49)), and the $\|Lu\|_0^2$ term can be absorbed into the left-hand side. The middle term in the last line of (4.103) is again estimated as above. It thus only remains to estimate the first term in the last line of (4.103). To this end, write

$$[L, \overline{L}] = aL + b\overline{L} + c(L_2 - \overline{L_2}), \tag{4.104}$$

near $b\Omega$, where $L_2 = (\partial\rho/\partial\bar{z}_1)(\partial/\partial z_1) + (\partial\rho/\partial\bar{z}_2)(\partial/\partial z_2)$ is the complex normal to the level sets of ρ. Actually, $b = -\bar{a}$ (since $[L, \overline{L}] = -\overline{[L, \overline{L}]}$), but this is not needed here. By what was said above, the contributions in (4.103) form all the terms in (4.104) are either under control or can be absorbed, with the exception of the one coming from cL_2. For its contribution, we have the estimate (again by integration by parts)

$$\int_\Omega cL_2u\,\bar{u} = -\int_\Omega cu\,\overline{\overline{L_2}u} + \int_{b\Omega} c|u|^2\left(\frac{L_2\rho}{|\nabla\rho|}\right) + O(\|u\|^2). \tag{4.105}$$

The first and the last term on the right-hand side of (4.105) are $O\left(\|u\|\|\overline{L_2}u\|\right)$ and $O\left(\|u\|^2\right)$, respectively. By what was said above, they are under control. For the boundary term, we note that c is 'the' Levi form applied to L, $\overline{L}$. Because $u \in \mathrm{dom}(\bar{\partial}^*)$, the vector (u_1, u_2) is complex tangential at points of the boundary; thus $(u_1, u_2) = |u|L$. Consequently, $c|u|^2$ is comparable to $\sum_{j,k=1}^2(\partial^2\rho/\partial z_j\partial\bar{z}_k)u_j\overline{u_k}$, and the integral is comparable to $\int_{b\Omega}\sum_{j,k=1}^2(\partial^2\rho/\partial z_j\partial\bar{z}_k)u_j\overline{u_k}$. But this integral is also dominated by $\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2$, again by the Kohn–Morrey formula. (4.104) holds only near $b\Omega$. But on a relatively compact subset V of Ω, $\|[L, \overline{L}]u\|^2 \lesssim \|u\|_{W^1(V)}^2$ is dominated by $\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2$, by interior elliptic regularity (see (2.85)). This completes the proof of Lemma 4.32. □

5 Regularity in Sobolev spaces

In the last chapter, we have seen that the canonical solution operator $\bar{\partial}^* N_q$ for $\bar{\partial}$ satisfies estimates in Sobolev norms on a smooth bounded pseudoconvex domain Ω when N_q is compact as an operator on $\mathcal{L}^2_{(0,q)}(\Omega)$. In this chapter, we first address the question of solving $\bar{\partial}$ with Sobolev estimates on a general domain. Sobolev estimates for the $\bar{\partial}$-Neumann operator need not be available ([11], [83], see Theorem 5.21 below). However, work of Kohn [191] shows that for $k \in \mathbb{N}$ fixed, there is a weight $e^{-\varphi}$ so that the weighted canonical solution operator $\bar{\partial}^*_{\varphi} N_{\varphi,q}$ (see (2.21) through (2.23)) is continuous in Sobolev-k norms. With some additional work, one can combine these operators to obtain a linear solution operator that is continuous on $C^{\infty}(\overline{\Omega})$. Another application of the regularity results for the weighted operators gives the precise relationship between global regularity properties of the $\bar{\partial}$-Neumann operators N_q and the Bergman projections P_q.

We then return to the unweighted $\bar{\partial}$-Neumann problem and study regularity in Sobolev spaces. It turns out that there are many domains where compactness fails, but regularity in Sobolev spaces holds. For example, global regularity holds on all convex domains, regardless of whether or not there are analytic varieties in the boundary (whereas compactness holds only when there are no varieties in the boundary, by Theorem 4.26). On the other hand, this regularity does not hold on all domains: work of Christ [83] shows that on the so called worm domains in $\mathbb{C}^2$, N_1 does not even preserve $C^{\infty}(\overline{\Omega})$. The positive results are obtained via families of vector fields that have good approximate commutation properties with $\bar{\partial}$. The existence of these families is equivalent to a certain 1-form α on the boundary being approximately $\bar{\partial}$-exact on the null space of the Levi form via $\bar{\partial}$ of approximately real functions (see Proposition 5.13 for a precise formulation).

The chapter concludes with a recent result from [287] that shows how the approaches to global regularity via compactness and via vector fields having favorable commutation properties with $\bar{\partial}$ can be unified under one general theory. It is noteworthy that the sufficient condition for global regularity that arises discriminates among form levels: it becomes progressively weaker as the level q increases (as in the (P_q) and $(\widetilde{P_q})$ conditions for compactness). In particular, when $q > 1$, one obtains results that are not covered by the vector field method.

5.1 Kohn's weighted theory

Assume that Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. For $t \geq 0$, let $\varphi_t(z) = t|z|^2$. Note that since $e^{-\varphi_t}$ is bounded and bounded away from zero on $\overline{\Omega}$, the weighted and unweighted norms are equivalent. Recall from Chapter 1 that

$\mathrm{dom}(\bar{\partial}^*) = \mathrm{dom}(\bar{\partial}^*_{\varphi_t})$, and that for $u = \sum'_{|J|=q} u_J d\bar{z}_J$ in this common domain

$$\bar{\partial}^*_t u = \bar{\partial}^* u + \sum_{j=1}^{n} \sum_{|K|=q-1}{}' \frac{\partial \varphi_t}{\partial z_j} u_{jK}\, d\bar{z}_K,\ 1 \le q \le n\,. \tag{5.1}$$

For economy of notation, we use subscripts t instead of φ_t (which would correspond to the convention used in Chapter 1). In view of (5.1), Proposition 2.3 gives that forms smooth up to the boundary are dense in the various graph norms in the weighted setup as well. Combining this with Proposition 2.4 with $a \equiv 0$ and $\varphi = \varphi_t$ (so that $\partial^2\varphi/\partial z_j \partial\bar{z}_k = t\delta_{j,k}$), we obtain

$$tq\|u\|_t^2 \le \|\bar{\partial} u\|_t^2 + \|\bar{\partial}^*_t u\|_t^2,\ u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*_t); \tag{5.2}$$

the factor q arises as in (2.49): every strictly increasing multi-index J of length q arises q times in the relevant sum in (2.24).

With (5.2) in hand, we can retrace the steps in Chapter 1, starting with Proposition 2.7; its role is now played by (5.2). We obtain that $\Box_{t,q} := \bar{\partial}\bar{\partial}^*_t + \bar{\partial}^*_t\bar{\partial}$ has a bounded inverse on $\mathcal{L}^2_{(0,q)}(\Omega, e^{-\varphi_t})$. This inverse is the weighted $\bar{\partial}$-Neumann operator $N_{t,q}$. Proposition 2.8, Theorem 2.9 and Corollary 2.10 remain valid in the weighted context, with the same proofs as in the unweighted case. The factor t in the left-hand side of (5.2) is crucial; it makes it possible to prove the following result from [191].

Theorem 5.1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, and $k \in \mathbb{N}$. Then there exists $t_0 \ge 0$ such that for all $t \ge t_0$, the operators $N_{t,q}$, $\bar{\partial}N_{t,q}$, $\bar{\partial}^*_t N_{t,q}$, $\bar{\partial}^*_t\bar{\partial}N_{t,q}$, $\bar{\partial}\bar{\partial}^*_t N_{t,q}$, and $P_{t,q-1}$, are all continuous in $W^k(\Omega)$ for $1 \le q \le n$.*

Remark. The various operators in Theorem 5.1 are all continuous in $\mathcal{L}^2(\Omega)$. By interpolation, they are thus continuous in $W^s(\Omega)$ for $0 \le s \le k$.

Proof of Theorem 5.1. We first consider $N_{t,q}$. Recall the second remark made after the proof of Theorem 4.6: to obtain estimates for a fixed Sobolev level k (rather than for all k), a compactness estimate is only needed with some small enough $\varepsilon = \varepsilon(k)$ (see (4.28)). The role of the compactness estimate is now played by (5.2). The argument follows that in the proof of Theorem 4.6 essentially line by line (and thus it follows [201], [191]); with all the operators replaced by their weighted counterparts. This includes the regularized operators $N_{t,\delta,q}$; they are defined via the Hermitian form $Q_{t,\delta}(u,v) = (\bar{\partial}u, \bar{\partial}v)_t + (\bar{\partial}^*_t u, \bar{\partial}^*_t v)_t + \delta(\nabla u, \nabla v)_t$, for $u, v \in W^1_{(0,q)}(\Omega) \cap \mathrm{dom}(\bar{\partial}^*_t)$. Choosing t big enough (i.e., $1/t$ small enough) allows one to absorb the relevant terms. Consequently, one has to keep track of the dependence of the various constants on t; this results in a modification of the crucial estimate (3.50), and its proof, which we now indicate (compare also [191]). With the notation from (3.50), we claim that for $k \in \mathbb{N}$, there exist a constant C independent of t and of δ, and constants C_t independent of δ, such that

$$Q_{t,\delta}(T^\alpha N_{t,\delta,q} u, T^\alpha N_{t,\delta,q} u) \le C\left(\|u\|_{t,k}^2 + \|N_{t,\delta,q} u\|_{t,k}^2\right) + C_t\|u\|_t^2. \tag{5.3}$$

(5.3) is proved by induction on k. When $k = 0$, it holds by definition of $N_{t,\delta,q}$. So assume now that it holds up to some integer $k-1 \geq 0$; we have to show that it holds for k. To do this, we can follow (3.55) through (3.58), paying attention to the dependence of constants on t. (3.55) is modified to

$$\begin{aligned} &Q_{t,\delta}(T^\alpha N_{t,\delta,q}u, T^\alpha N_{t,\delta,q}u) \\ &\quad = (T^\alpha \bar{\partial} N_{t,\delta,q}u, \bar{\partial} T^\alpha N_{t,\delta,q}u)_t + (T^\alpha \bar{\partial}_t^* N_{t,\delta,q}u, \bar{\partial}_t^* T^\alpha N_{t,\delta,q}u)_t \\ &\qquad + \delta(T^\alpha \nabla N_{t,\delta,q}u, \nabla T^\alpha N_{t,\delta,q}u)_t + \frac{1}{2} Q_{t,\delta}(T^\alpha N_{\delta,q}u, T^\alpha N_{\delta,q}u) \\ &\qquad + O(\|N_{\delta,q}u\|_{t,k}^2) + O_t\big(\|N_{t,\delta,q}u\|_{t,k-1}^2\big). \end{aligned} \tag{5.4}$$

Here, O indicates a constant that does not depend on t nor on δ, whereas O_t indicates a constant that is allowed to depend on t, but not on δ. We have used that the k-th order term in $[\bar{\partial}_t^*, T^\alpha]$ is independent of t (recall that $\bar{\partial}^*$ and $\bar{\partial}_t^*$ differ by an operator of order zero). The $(1/2)Q_{t,\delta}$-term is absorbed as in Chapter 2, and the second to the last term on the right-hand side of (5.4) is dominated by the right-hand side of (5.3). The last term can be estimated by interpolation of Sobolev norms by $\|N_{t,\delta,q}u\|_{t,k}^2 + C_t\|N_{t,\delta,q}u\|_t^2 \leq \|N_{t,\delta,q}u\|_{t,k}^2 + C_t\|u\|_t^2$, and so is also dominated by the right-hand side of (5.3). (We have also used that $\|N_{t,\delta,q}u\|_t \leq (1/tq)\|u\|_t$, which follows from (5.2).) For the remaining terms on the right-hand side of (5.4), we have the analogue of (3.56):

$$\begin{aligned} &(\bar{\partial} N_{t,\delta,q}u, (T^\alpha)_t^* \bar{\partial} T^\alpha N_{t,\delta,q}u)_t + (\bar{\partial}_t^* N_{t,\delta,q}u, (T^\alpha)_t^* \bar{\partial}_t^* T^\alpha N_{t,\delta,q}u)_t \\ &\quad + \delta(\nabla N_{t,\delta,q}u, (T^\alpha)_t^* \nabla T^\alpha N_{t,_t\delta,q}u)_t. \end{aligned} \tag{5.5}$$

Then, as in (3.57),

$$\begin{aligned} (\bar{\partial} N_{t,\delta,q}u, (T^\alpha)_t^* \bar{\partial} T^\alpha N_{t,\delta,q}u)_t = {} & (\bar{\partial} N_{t,\delta,q}u, \big[[(T^\alpha)_t^*, \bar{\partial}], T^\alpha\big] N_{t,\delta,q}u)_t \\ & + (\bar{\partial} N_{t,\delta,q}u, T^\alpha[(T^\alpha)_t^*, \bar{\partial}] N_{t,\delta,q}u)_t \\ & + (\bar{\partial} N_{t,\delta,q}u, \bar{\partial}(T^\alpha)_t^* T^\alpha N_{t,\delta,q}u)_t. \end{aligned} \tag{5.6}$$

We analyze the first term on the right-hand side of (5.6) along the lines of the argument for the corresponding term in (3.57). Because $(T^\alpha)_t^*$ differs from $(T^\alpha)^*$ by an operator of order not exceeding $k-1$, the result is that this term is $O(\|N_{t,\delta,q}u\|_{t,k}^2) + O_t(\|N_{t,\delta,q}u\|_{t,k-1}^2)$, hence is controlled by the right-hand side of (5.3) (as in (5.4) above, via interpolation of Sobolev norms).

In the second term on the right-hand side of (5.6), we integrate T^α by parts to the left as $(T^\alpha)_t^* = T^\alpha + A_t$, where A_t is a tangential operator of order at most $k-1$, with coefficients that depend on t. Commuting both T^α and A_t with $\bar{\partial}$ and using that $(T^\alpha)_t^*$ differs from $(T^\alpha)^*$ by an operator of order at most $k-1$, we can argue as in (3.58) to obtain that this term equals

$$\begin{aligned} &(\bar{\partial} T^\alpha N_{t,\delta,q}u, [(T^\alpha)_t^*, \bar{\partial}] N_{t,\delta,q}u)_t + t(\bar{\partial} S^\beta N_{t,\delta,q}u, [(T^\alpha)_t^*, \bar{\partial}] N_{t,\delta,q}u)_t \\ &\quad + O\big(\|N_{t,\delta,q}u\|_{t,k}^2\big) + O_t\big(\|N_{t,\delta,q}u\|_{t,k-1}\|N_{t,\delta,q}u\|_{t,k}\big), \end{aligned} \tag{5.7}$$

where S^β denotes a tangential operator of order not more than $k-1$. Using again that $(T^\alpha)^*_t$ differs from $(T^\alpha)^*$ by an operator of order at most $k-1$ (for the first term), the induction hypothesis (for the second term), and the inequality $|ab| \leq \text{l.c.}|a|^2 + \text{s.c.}|b|^2$, (5.7) is dominated by

$$\begin{aligned} &\text{s.c.}\|\bar\partial T^\alpha N_{t,\delta,q}u\|^2_t + \text{l.c.}\|N_{t,\delta,q}u\|^2_{t,k} \\ &\quad + O\big(\|N_{t,\delta,q}u\|^2_{t,k}\big) + O_t\big(\|N_{t,\delta,q}u\|^2_{t,k-1} + \|u\|^2_{t,k-1}\big), \end{aligned} \tag{5.8}$$

where l.c. and s.c. are independent of t, δ. The first term can be absorbed into the left-hand side of (5.3), the second term can go into the third, which is dominated by the right-hand side of (5.3), as is the last term (again by interpolation).

We are left with only the third term in (5.6). This was for the first term in (5.5), but similar arguments apply to the other two terms in (5.5), and we are left with the sum of the corresponding third terms in (5.6), i.e., with

$$\begin{aligned} Q_{t,\delta}(N_{t,\delta,q}u, (T^\alpha)^*_t T^\alpha N_{t,\delta,q}u) &= (u, (T^\alpha)^*_t T^\alpha N_{t,\delta,q}u)_t \\ &= (T^\alpha u, T^\alpha N_{t,\delta,q}u)_t \\ &= O\big(\|u\|^2_{t,k} + \|N_{t,\delta,q}u\|^2_{t,k}\big). \end{aligned} \tag{5.9}$$

This completes the proof of (5.3), and, by what was said above, the proof of the estimates for $N_{t,q}$.

Once the estimates for $N_{t,q}$ are in place, those for $\bar\partial N_{t,q}$, $\bar\partial^*_t N_{t,q}$, $\bar\partial\bar\partial^*_t N_{t,q}$, and $\bar\partial^*_t\bar\partial N_{t,q}$ follow as in the unweighted case. It suffices to show the estimates for $u \in C^\infty_{(0,q)}(\overline{\Omega})$ (since this space is dense in $W^k_{(0,q)}(\Omega)$). Now the desired estimates follow essentially from (the proof of) Lemma 3.2, with the following modification. The proof of Lemma 3.2 proceeds by absorbing terms; that they are finite is guaranteed by the assumption that $N_q u \in C^\infty_{(0,q)}(\overline{\Omega})$. In our case, we only know that $N_{t,q}u \in W^k_{(0,q)}(\Omega)$, hence that $\bar\partial N_{t,q}$, $\bar\partial^*_t N_{t,q}$, $\bar\partial\bar\partial^*_t N_{t,q}$, and $\bar\partial^*_t\bar\partial N_{t,q}$, are in $W^{k-1}(\Omega)$ and $W^{k-2}(\Omega)$, respectively. However, this situation is easily rectified. It suffices to choose t big enough so that $N_{t,q}$ is continuous in $W^{k+2}_{(0,q)}(\Omega)$. Then all forms in question are (at least) in $W^k(\Omega)$ (for $u \in C^\infty_{(0,q)}(\overline{\Omega})$), and the argument can proceed as in the proof of Lemma 3.2.

In view of the formula $P_{t,q} = \bar\partial\bar\partial^*_t N_{t,q}$ (see (2.70) and (2.71)), we have the estimates that are claimed in Theorem 5.1 for the Bergman projections $P_{t,q}$ for $1 \leq q \leq n$, but not for $P_{t,0}$. The proof that $P_{t,0}$ nonetheless is continuous in $W^k(\Omega)$ (t big enough) follows the argument used in [194] to show that compactness of N_1 implies Sobolev estimates for P_0. The type of argument used in these proofs is by now familiar enough that we will only give the proof for $k = 1$. That is, we will prove: there exists t_0 such that for $t \geq t_0$, the weighted Bergman projection $P_{t,0}$ maps $W^1(\Omega)$ continuously to itself. Let $u \in C^\infty(\overline{\Omega})$. By what we already proved, $P_{t,0}u = u - \bar\partial^*_t N_{t,1}\bar\partial u$ is in $W^1(\Omega)$ when t is big enough so that $N_{t,1}$ is continuous on $W^2_{(0,1)}(\Omega)$). In particular, $\|P_{t,0}u\|_{t,1}$ is finite. By the Cauchy–Riemann equations for the holomorphic function

$P_{t,0}u$, we only need to estimate tangential derivatives in order to estimate the norm in $W^1(\Omega)$. Consider T as in the proof above (i.e., as in the proof of Theorem 4.6); in particular, T acts componentwise in a special boundary chart, and so preserves the domain of $\bar\partial_t^*$. Then $\|TP_{t,0}u\|_t^2 \le \|Tu\|_t^2 + \|T\bar\partial_t^* N_{t,1}\bar\partial u\|_t^2$, and

$$\begin{aligned}(T\bar\partial_t^* N_{t,1}\bar\partial u, T\bar\partial_t^* N_{t,1}\bar\partial u)_t &\\ &= ([T, \bar\partial_t^*]N_{t,1}\bar\partial u, T\bar\partial_t^* N_{t,1}\bar\partial u)_t + (TN_{t,1}\bar\partial u, \bar\partial T\bar\partial_t^* N_{t,1}\bar\partial u)_t \\ &= ([T, \bar\partial_t^*]N_{t,1}\bar\partial u, T\bar\partial_t^* N_{t,1}\bar\partial u)_t + (TN_{t,1}\bar\partial u, [\bar\partial, T]\bar\partial_t^* N_{t,1}\bar\partial u)_t \\ &\quad + (TN_{t,1}\bar\partial u, T\bar\partial\bar\partial_t^* N_{t,1}\bar\partial u)_t.\end{aligned} \tag{5.10}$$

The first order term of $[T, \bar\partial_t^*]$ is independent of t, $N_{t,1}\bar\partial$ is continuous in $\mathcal{L}^2(\Omega)$, and $\bar\partial\bar\partial_t^* N_{t,1}\bar\partial u = \bar\partial u$. Therefore, the right-hand side of (5.10) is estimated from above by

$$\text{s.c.}\|\bar\partial_t^* N_{t,1}\bar\partial u\|_{t,1}^2 + \text{l.c.}\|N_{t,1}\bar\partial u\|_{t,1}^2 + C_t\|u\|_t^2 + (TN_{t,1}\bar\partial u, T\bar\partial u)_t, \tag{5.11}$$

where s.c. and l.c. are independent of t. In the last term in (5.11), we perform the steps in (5.10) in the reverse order to obtain

$$\begin{aligned}(TN_{t,1}\bar\partial u, T\bar\partial u)_t &= (T\bar\partial_t^* N_{t,1}\bar\partial u, Tu)_t \\ &\quad + \big([\bar\partial_t^*, T]N_{t,1}\bar\partial u, Tu\big)_t + O\big(\|TN_{t,1}\bar\partial u\|_t\|u\|_{t,1}\big);\end{aligned} \tag{5.12}$$

this term is thus dominated, independently of t, by the first two terms in (5.11) plus $C_t\|u\|_{t,1}^2$. We next estimate $\|N_{t,1}\bar\partial u\|_{t,1}^2$. Modulo terms that are under control, it again suffices to estimate $\|TN_{t,1}\bar\partial u\|_t^2$. (5.2) gives

$$\begin{aligned}\|TN_{t,1}\bar\partial u\|_t^2 &\le \frac{1}{t}\big(\|\bar\partial TN_{t,1}\bar\partial u\|_t^2 + \|\bar\partial_t^* TN_{t,1}\bar\partial u\|_t^2\big) \\ &\le \frac{C}{t}\big(\|N_{t,1}\bar\partial u\|_{t,1}^2 + C_t\|u\|_t^2 + \|T\bar\partial_t^* N_{t,1}\bar\partial u\|_t^2\big).\end{aligned} \tag{5.13}$$

The first two terms inside the parentheses on the right-hand side of (5.13) arise from the commutators $[\bar\partial, T]$ and $[\bar\partial_t^*, T]$. Summing over a suitable collection of T's, using (2.85) to estimate norms on compact subsets, and choosing t big enough, we find after absorbing terms

$$\|N_{t,1}\bar\partial u\|_{t,1}^2 \le \frac{C}{t}\|\bar\partial_t^* N_{t,1}\bar\partial u\|_{t,1}^2 + C_t\|u\|_t^2. \tag{5.14}$$

We similarly sum in (5.10). Choosing the constant s.c. in (5.11) small enough to absorb $\|\bar\partial_t^* N_{t,1}\bar\partial u\|_{t,1}^2$, using (5.13) for the last term in (5.12), and substituting estimate (5.14) for $\|N_{t,1}\bar\partial u\|_{t,1}^2$, gives

$$\|\bar\partial_t^* N_{t,1}\bar\partial u\|_{t,1}^2 \le \frac{C}{t}\|\bar\partial_t^* N_{t,1}\bar\partial u\|_{t,1}^2 + C_t\|u\|_{t,1}^2. \tag{5.15}$$

Choosing t big enough one more time gives the desired estimate for $\bar{\partial}_t^* N_{t,1}\bar{\partial}u$, hence for $P_{t,0}$.

The proof of Theorem 5.1 is now complete. □

Remarks. (i) We will also require Sobolev estimates for the operators $N_{t,q}\bar{\partial}$, where $1 \le q \le n$. When $q > 1$, these estimates are immediate from Theorem 5.1 and the commutation relation $N_{t,q}\bar{\partial} = \bar{\partial}N_{t,q-1}$ on $\mathrm{dom}(\bar{\partial})$ (the latter is analogous to part 3 in Theorem 2.9; see also Remark (ii) following the statement of Theorem 2.9). The Sobolev estimates for $N_{t,1}\bar{\partial}$ result from (5.14) above and the corresponding estimates for $\bar{\partial}_t^* N_{t,1}\bar{\partial}$.

(ii) Theorem 5.1 says that by switching to a suitable weighted norm, the behavior with respect to Sobolev estimates of the $\bar{\partial}$-Neumann and related operators can be improved. In view of (5.1), this is not the case for a subelliptic or a compactness estimate. It is less obvious that subellipticity and compactness are independent of the metric (say among metrics smooth and nondegenerate on the closure) even when general metrics on q-forms are allowed (rather than one that is, at each point, just a multiple of the Euclidean metric). For a general result in the coercive case, see [292]; a simple proof for subellipticity and for compactness in the case of the $\bar{\partial}$-Neumann problem is in [75].

(iii) Theorem 5.1 implies that the scale of Sobolev spaces $W^s_{(0,q)}(\Omega) \cap \ker(\bar{\partial})$, $s \ge 0$, forms a so called interpolation scale. This observation is from [39]; for general information on interpolation spaces, see [33]. Whether this property holds in general, without the assumption of pseudoconvexity, seems to be open. For results on starshaped domains, via a different approach, see [282].

(iv) Solvability with Sobolev estimates, but with a loss of three derivatives, has been obtained by S. L. Yie ([303]) for domains with only C^4 boundary.

(v) For a version of Theorem 5.1 valid when the boundary is only C^k, $k \ge 1$, see [160].

Theorem 5.1 has the following useful corollary.

Corollary 5.2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, s_1, s_2 both nonnegative, $s_1 \le s_2$. Then $W^{s_2}_{(0,q)}(\Omega) \cap \ker(\bar{\partial})$ is dense in $W^{s_1}_{(0,q)}(\Omega) \cap \ker(\bar{\partial})$.*

Proof. We use that $W^{s_2}_{(0,q)}(\Omega)$ is dense in $W^{s_1}_{(0,q)}(\Omega)$. So given $u \in W^{s_1}_{(0,q)}(\Omega) \cap \ker(\bar{\partial})$, there is a sequence of u_j's in $W^{s_2}_{(0,q)}(\Omega)$ that converges to u in the norm of $W^{s_1}_{(0,q)}(\Omega)$. Choose t big enough so that P_t is continuous on $W^{s_2}_{(0,q)}(\Omega)$ (and hence in $W^{s_1}_{(0,q)}(\Omega)$ by the remark above). Then $P_t u_j \in W^{s_2}_{(0,q)}(\Omega) \cap \ker(\bar{\partial})$, and $P_t u_j \to P_t u = u$ in the norm of $W^{s_1}_{(0,q)}(\Omega)$. □

Remarks. (i) Corollary 5.2 and Theorem 5.3 below will combine to show that even $C^\infty_{(0,q)}(\overline{\Omega})$ is dense in $W^s_{(0,q)}(\Omega)$, $s \ge 0$, see Corollary 5.4.

(ii) Corollary 5.2 may fail when the domain is not assumed pseudoconvex; for examples, see [9], [13], [49].

5.2 A linear solution operator continuous in $C^\infty(\overline{\Omega})$

Suppose that $u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap \ker(\bar\partial)$ is given. By Theorem 5.1, there exists, for each $k \in \mathbb{N}$, $v_k \in W^k_{(0,q-1)}(\Omega)$ with $\bar\partial v_k = u$, namely $v_k = \bar\partial^*_{t_k} N_{t_k} u$ with t_k chosen sufficiently big. This together with Corollary 5.2 allows for a Mittag-Leffler type construction (see below) that results in a solution v smooth up to the boundary. This observation is in [192] (p. 230), where it is credited to a communication from L. Hörmander. With some additional work, one can obtain a solution operator that is continuous in Sobolev spaces with arbitrarily small loss. Whether there is such an operator that is exactly regular in Sobolev spaces is open.

$C^\infty(\overline{\Omega})$ has the usual Fréchet space topology induced by $C^k(\overline{\Omega})$, $k \in \mathbb{N}$, or equivalently (by the Sobolev embedding theorem) by $W^k(\Omega)$, $k \in \mathbb{N}$. That is, a sequence converges in $C^\infty(\overline{\Omega})$ if and only if it converges in $W^k(\Omega)$ for all $k \in \mathbb{N}$. An operator T on $C^\infty(\overline{\Omega})$ is continuous if and only if for each $k_1 \in \mathbb{N}$, there exists $k_2 \in \mathbb{N}$ and a constant C_{k_1,k_2} such that $\|Tu\|_{k_1} \le C_{k_1,k_2}\|u\|_{k_2}$, $u \in C^\infty(\overline{\Omega})$. (For information on Fréchet spaces, see for example [298], [260].)

Theorem 5.3. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n$. There exists a continuous linear operator $T_q : C^\infty_{(0,q)}(\overline{\Omega}) \cap \ker(\bar\partial) \to C^\infty_{(0,q-1)}(\overline{\Omega})$ such that $\bar\partial T_q u = u$; moreover, T_q satisfies $\|T_q u\|_{j-\delta} \le C_{j,\delta}\|u\|_j$ for $j = 1, 2, \dots,$ and $\delta > 0$.*

Proof. Let $u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap \ker(\bar\partial)$. We first construct $w \in C^\infty_{(0,q-1)}(\overline{\Omega})$ with $\bar\partial w = u$, without paying attention to linearity, to see what the basic idea (i.e., the Mittag-Leffler type construction) is. As mentioned above, Theorem 5.1 provides a sequence $(w_k)_1^\infty$ with $w_k \in W^k_{(0,q-1)}(\Omega)$ and $\bar\partial w_k = u$. Note that one can modify w_k by a $\bar\partial$-closed form in $W^k_{(0,q-1)}(\Omega)$ without affecting the two properties in the previous sentence. The idea is now to modify the sequence to obtain one where $\|w_{m+1} - w_m\|_m \le 1/2^m$, which will result in a limit in $C^\infty_{(0,q-1)}(\overline{\Omega})$. We do this inductively. w_1 stays the same: $\widehat{w_1} = w_1$. $w_2 - w_1$ is $\bar\partial$-closed and belongs to $W^1_{(0,q-1)}(\Omega)$. By Corollary 5.2, there is $v \in W^2_{(0,q-1)}(\Omega) \cap \ker(\bar\partial)$ with $\|v - (w_2 - w_1)\|_1 \le 1/2$. Set $\widehat{w_2} = w_2 - v$. Then $\widehat{w_2} \in W^2_{(0,q-1)}(\Omega)$, $\bar\partial\widehat{w_2} = u$, and $\|\widehat{w_2} - \widehat{w_1}\|_1 = \|w_2 - w_1 - v\|_1 \le 1/2$. Proceeding inductively, approximating $w_{m+1} - \widehat{w_m}$ to within $1/2^m$ at the m-th step, we obtain a sequence $(\widehat{w_m})_1^\infty$ with the desired properties. Now we set

$$w = \widehat{w_1} + \sum_{m=1}^{\infty} \left(\widehat{w_{m+1}} - \widehat{w_m}\right). \tag{5.16}$$

(5.16) converges in $W^1_{(0,q-1)}(\Omega)$, since $\|\widehat{w_{m+1}} - \widehat{w_m}\|_1 \le \|\widehat{w_{m+1}} - \widehat{w_m}\|_m \le 1/2^m$. But for any $k \in \mathbb{N}$, $w = \widehat{w_k} + \sum_{m=k}^{\infty}\left(\widehat{w_{m+1}} - \widehat{w_m}\right)$, and this sum converges in W^k (again because $\|\widehat{w_{m+1}} - \widehat{w_m}\|_k \le \|\widehat{w_{m+1}} - \widehat{w_m}\|_m \le 1/2^m$ when $m \ge k$). Consequently, $w \in W^k_{(0,q-1)}(\Omega)$ for all k, and hence in $C^\infty_{(0,q-1)}(\overline{\Omega})$, by the Sobolev embedding theorem. It is clear that $\bar\partial w = u$.

In order to obtain a linear solution operator, the approximation (from Corollary 5.2) needs to be done via linear operators. We use a construction that uses the same type of mollifiers that were used in the proof of part (ii) of Proposition 2.3 (and that were taken from [170]). Cover the boundary of Ω by finitely many open sets U_j, $j = 1, \dots, N$ so that there are truncated open cones Γ_j such that $z - \zeta \in \Omega$ when $z \in U_j \cap \overline{\Omega}$ and $\zeta \in \Gamma_j$. Denote by U_0 an open relatively compact subset of Ω such that $\overline{\Omega} \subset \bigcup_{j=0}^N U_j$. Next choose $\varphi_j \in C_0^\infty(U_j)$, $0 \le j \le N$, such that $\sum_0^N \varphi_j \equiv 1$ in a neighborhood of $\overline{\Omega}$. Denote by Γ_0 the 'cone' $\{|\zeta| < a\}$, where a is small enough so that $z - \zeta \in \Omega$ when $z \in U_0$ and $|\zeta| < a$. Finally, choose $\chi_j \in C_0^\infty(\Gamma_j)$, $0 \le j \le N$, with $\chi_j \ge 0$ and $\int \chi_j = 1$. For $0 < \varepsilon < 1$, set $\chi_{j,\varepsilon} = \varepsilon^{-2n}\chi_j(z/\varepsilon)$. Note that then $\chi_{j,\varepsilon} \in C_0^\infty(\Gamma_j)$ as well.

For a function $h \in \mathcal{L}^2(\Omega)$ and $0 < \varepsilon < 1$, define

$$S^\varepsilon h(z) = \sum_{j=1}^N \chi_{j,\varepsilon} * \varphi_j h, \tag{5.17}$$

where $\left(\chi_{j,\varepsilon} * \varphi_j h\right)(z) = \int_{\mathbb{C}^n} \chi_{j,\varepsilon}(\zeta)\left(\varphi_j h\right)(z-\zeta)dV(\zeta)$. Then $S^\varepsilon \colon \mathcal{L}^2(\Omega) \to C^\infty(\overline{\Omega})$ is continuous (see also the proof of part (ii) of Proposition 2.3). We define S^ε on forms coefficientwise.

Via a variant of the Mittag-Leffler argument from above we construct inductively a sequence of continuous linear operators $R_k \colon W^k_{(0,q)}(\Omega) \to W^k_{(0,q-1)}(\Omega)$ such that (i) R_k is continuous $W^m_{(0,q)}(\Omega) \to W^m_{(0,q-1)}(\Omega)$, $0 \le m \le k$ (ii) $\bar\partial R_k u = u$, and (iii) $\|R_{k+1}u - R_k u\|_{m-\delta} \le (C/k^2)\|u\|_m$, $1 \le m \le k$ (for $\delta > 0$ fixed). To start, set $R_1 u = \bar\partial_t^* N_t u$ for $u \in W^1_{(0,q)}(\Omega)$, where t is chosen big enough so that R_1 is continuous in W^1-norms. Assume $R_1, \dots, R_k$ have been constructed, for some $k \ge 1$. To construct R_{k+1}, set, for $u \in W^{k+1}_{(0,q)}(\Omega)$,

$$R_{k+1}u = S^\varepsilon R_k u + \bar\partial_t^* N_{t,q}\left(u - \bar\partial S^\varepsilon R_k u\right), \tag{5.18}$$

where now t is chosen big enough so that $\bar\partial_t^* N_{t,q}$ is continuous in $W^{k+1}(\Omega)$ and ε (small) is to be specified later. The first term on the right-hand side of (5.18) approximates $R_k u$ by a form in $C^\infty(\overline{\Omega})$; the second term then makes a correction to ensure that $\bar\partial R_{k+1}u = u$ (thus we have (ii)). In view of the mapping properties of S^ε and of $\bar\partial_t^* N_{t,q}$, we have for $u \in W^{k+1}_{(0,q)}(\Omega)$ and $0 \le m \le k+1$:

$$\begin{aligned}\|R_{k+1}u\|_m &\le C_{k,\varepsilon}\left(\|R_k u\|_{m-1} + \|u - \bar\partial S^\varepsilon R_k u\|_m\right)\\ &\le C_{k,\varepsilon}\left(\|u\|_{m-1} + \|u\|_m + \|S^\varepsilon R_k u\|_{m+1}\right)\\ &\le C_{k,\varepsilon}\|u\|_m + C_{k,\varepsilon}\|R_k u\|_{m-1} \le C_{k,\varepsilon}\|u\|_m.\end{aligned} \tag{5.19}$$

(As usual, the constant $C_{k,\varepsilon}$ is allowed to change its value from one occurrence to the next.) This verifies (i).

To prove (iii), let $1 \leq m \leq k$. We have

$$\|R_{k+1}u - R_k u\|_{m-1} \leq \|S^{\varepsilon} R_k u - R_k u\|_{m-1} + \|\bar{\partial}_t^* N_t \left(u - \bar{\partial} S^{\varepsilon} R_k u\right)\|_{m-1}. \quad (5.20)$$

To estimate the second term on the right-hand side of (5.20), it suffices to estimate $\|u - \bar{\partial} S^{\varepsilon_k} R_k u\|_{m-1}$ (since $\bar{\partial}_t^* N_t$ is continuous in W^s-norms for $0 \leq s \leq (k+1)$). Note that on Ω, we may compute $\bar{\partial}$ first, and then convolve (because of the special choice of mollifiers, there are no boundary terms arising; see again the proof of part (ii) of Proposition 2.3). Thus

$$\bar{\partial} S^{\varepsilon} R_k u = \bar{\partial}\Big(\sum_{j=0}^{N} \chi_{j,\varepsilon} * \varphi_j R_k u\Big) = \sum_{j=0}^{N} \chi_{j,\varepsilon} * \left(\varphi_j u + \bar{\partial}\varphi_j \wedge R_k u\right), \quad (5.21)$$

where we have used that $\bar{\partial} R_k u = u$. Therefore, using that $\sum_{j=0}^{N} \varphi_j = 1$ in a neighborhood of $\overline{\Omega}$, and consequently $\sum_{j=0}^{N} \bar{\partial}\varphi_j = 0$, we have

$$\begin{aligned} \bar{\partial} S^{\varepsilon} R_k u - u = &\sum_{j=0}^{N} \left(\chi_{j,\varepsilon} * \varphi_j u - \varphi_j u\right) \\ &+ \sum_{j=0}^{N} \left(\chi_{j,\varepsilon} * \left(\bar{\partial}\varphi_j \wedge R_k u\right) - \bar{\partial}\varphi_j \wedge R_k u\right). \end{aligned} \quad (5.22)$$

To estimate the right-hand side, we use the fact that the W^{m-1}-norm of a difference quotient of a function is bounded by a constant times the W^m-norm of the function ([124], Theorem 6.21, [122], Theorem 3, Section 5.8.2). Convolution of, say, $\varphi_j u$, with $\chi_{j,\varepsilon}$ amounts to a convex combination of forms all of which have W^{m-1}-distance from $\varphi_j u$ not exceeding $C_k \varepsilon \|\varphi_j u\|_m$. Consequently this convex combination, $\chi_{j,\varepsilon} * \varphi_j u$, also has W^{m-1}-distance from $\varphi_j u$ not exceeding $C_k \varepsilon \|\varphi_j u\|_m$. Therefore

$$\|u - \bar{\partial} S^{\varepsilon} R_k u\|_{m-1} \leq C_k \varepsilon \Big(\sum_{j=0}^{N} \|\varphi_j u\|_m + \sum_{j=0}^{N} \|\bar{\partial}\varphi_j \wedge R_k u\|_m\Big) \leq C_k \varepsilon \|u\|_m. \quad (5.23)$$

By the same argument, we have for the first term in (5.20)

$$\|S^{\varepsilon} R_k u - R_k u\|_{m-1} \leq C_k \varepsilon \|u\|_m. \quad (5.24)$$

Combining (5.20) through (5.24) gives the estimate

$$\|R_{k+1}u - R_k u\|_{m-1} \leq C_k \varepsilon \|u\|_m. \quad (5.25)$$

Because convolution with the $\chi_{j,\varepsilon}$'s maps $W^m(\Omega)$ continuously to itself with norm bounded independently of ε (this reduces to the case of $W^0(\Omega) = \mathcal{L}^2(\Omega)$, where it is a consequence of Young's inequality, see for example [124]), we also have from the formulas above

$$\|R_{k+1}u - R_k u\|_m \leq C_k \|u\|_m. \quad (5.26)$$

(5.25) and (5.26) combined with interpolation of Sobolev norms give

$$\|R_{k+1}u - R_k u\|_{m-\delta} \le \frac{1}{k^2}\|u\|_m + \widetilde{C}_k \varepsilon \|u\|_m . \tag{5.27}$$

Choosing $\varepsilon \le 1/(k^2\widetilde{C}_k)$ gives (iii) above and finishes the inductive construction.

We can now construct the operator T_q from the theorem. For $u \in C^\infty_{(0,q)}(\overline{\Omega})$, we set

$$T_q u = R_1 u + \sum_{k=1}^{\infty} (R_{k+1}u - R_k u)\,. \tag{5.28}$$

The series converges in $W^{1-\delta}_{(0,q-1)}(\Omega)$ (in view of (iii) above), so $T_q u$ is well defined as an element of $W^{1-\delta}_{(0,q-1)}(\Omega)$. Also, $\bar\partial T_q u = \bar\partial R_1 u = u$. To obtain the continuity properties of T_q claimed in the theorem, let $j \in \mathbb{N}$ and note that $T_q u = R_j u + \sum_{k=j}^{\infty}(R_{k+1}u - R_k u)$. Thus

$$\begin{aligned} \|T_q u\|_{j-\delta} &\le \|R_j u\|_{j-\delta} + \sum_{k=j}^{\infty} \|R_{k+1}u - R_k u\|_{j-\delta} \\ &\le \left(C_j + \sum_{k=j}^{\infty} \frac{C}{k^2}\right)\|u\|_j . \end{aligned} \tag{5.29}$$

This completes the proof of Theorem 5.3; the continuity in $C^\infty(\overline{\Omega})$ follows from the Sobolev estimates (5.29). □

Corollary 5.4. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n$. Then $C^\infty_{(0,q)}(\overline{\Omega}) \cap \ker(\bar\partial)$ is dense in $W^s_{(0,q)}(\Omega) \cap \ker(\bar\partial)$ for $s \ge 0$.*

Proof. Let $u \in W^s_{(0,q)}(\Omega) \cap \ker(\bar\partial)$; we may also assume that $q \le (n-1)$ (for $q = n$, the condition to be in $\ker(\bar\partial)$ is vacuous). Fix $\varepsilon > 0$ and $j \in \mathbb{N}$, $j > s$. By Corollary 5.2, there is $\hat u \in W^{j+1}_{(0,q)}(\Omega) \cap \ker(\bar\partial)$ with $\|u - \hat u\|_s < \varepsilon/2$. Because $C^\infty(\overline{\Omega})$ is dense in $W^{j+1}(\Omega)$, there is a sequence $\{u_n\}_1^\infty \subset C^\infty_{(0,q)}(\overline{\Omega})$ such that $\|u_n - \hat u\|_{j+1} \to 0$. Thus $\|\bar\partial u_n\|_j = \|\bar\partial(u_n - \hat u)\|_j \to 0$ as well. Denote by T_{q+1} the solution operator to $\bar\partial$ from Theorem 5.3 and note that it is continuous from $W^j_{(0,q+1)}(\Omega)$ to $W^s_{(0,q)}(\Omega)$. Then $(u_n - T_{q+1}\bar\partial u_n) \in C^\infty_{(0,q)}(\overline{\Omega}) \cap \ker(\bar\partial)$ and $\|(u_n - T_{q+1}\bar\partial u_n) - \hat u\|_s \le \|u_n - \hat u\|_s + \|T_{q+1}\bar\partial u_n\|_s \to 0$. Consequently, for n sufficiently large, $\|(u_n - T_{q+1}\bar\partial u_n) - u\|_s \le \varepsilon$. As ε was arbitrary, the proof of Corollary 5.4 is finished. □

Remarks. (i) It follows from interpolation that when $s \ge 1$, then $T_q\colon W^s_{(0,q)}(\Omega) \to W^{s-\delta}_{(0,q-1)}(\Omega)$ is continuous. The case $0 < s < 1$ is not covered because $j = 0$ was omitted in Theorem 5.3; this was done to avoid technicalities.

(ii) Corollary 5.4 is shown in [29] to hold also when $s < 0$ in the case $q = 0$.

(iii) Dufresnoy has shown in [121] that when no boundary regularity assumptions are made on Ω, but instead the condition is imposed that $\overline{\Omega}$ can be approximated from the outside in a nicely controlled way by pseudoconvex domains, then one also has solvability in $C^\infty(\overline{\Omega})$ (here, $C^\infty(\overline{\Omega})$ is interpreted in the sense of Whitney). In particular, this holds on convex domains. Recently, Harrington has shown that Dufresnoy's assumption holds on all C^1 domains whose boundary satisfies property(P_1) ([159]). However, the assumptions are not satisfied on all smooth domains. In fact, there are smoothly bounded pseudoconvex domains whose closure does not admit any neighborhood basis consisting of pseudoconvex domains (a so called Stein neighborhood basis), much less a particularly nice one. For these domains, usually referred to as 'worm domains', see [105], [130], [81], [43], [208]; we will encounter them also later in this chapter.

5.3 Regularity: the Bergmann projections vs the $\bar{\partial}$-Neumann operators

Kohn's weighted theory can also be used to analyze in detail the relationship between regularity properties of the Bergman projections P_q and those of the $\bar{\partial}$-Neumann operators N_q, $0 \leq q \leq n$. Recall the relevant terminology: an operator on $(0,q)$-forms is said to be globally regular if it maps $C^\infty_{(0,q)}(\overline{\Omega})$ continuously to itself, and exactly regular if it maps the Sobolev spaces $W^s_{(0,q)}(\Omega)$ continuously to themselves for $s \geq 0$. When the operator maps from one form level to another, the definitions are analogous. It is clear from the formulas in Theorem 2.9 that there are relationships between the regularity properties of N and P. For example, from Kohn's formula $P_{q-1} = \mathbb{1} - \bar{\partial}^* N_q \bar{\partial}$ (see (2.71)), it is obvious that if N_q is globally regular, then so is P_{q-1}. It is however not obvious from this formula that exact regularity of N_q should imply exact regularity of P_{q-1}, nor that regularity properties of the Bergman projections should imply corresponding properties for the $\bar{\partial}$-Neumann operators. Therefore, the following theorem from [47] is of interest.

Theorem 5.5. *Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^n$, $1 \leq q \leq n$. Then N_q is exactly regular if and only if the three Bergman projections P_{q-1}, P_q and P_{q+1} are exactly regular. The same equivalence holds with 'exactly regular' replaced by 'globally regular'.*

Here, P_{n+1} formally acts on the zero-dimensional space of $(0, n+1)$-forms, so is always exactly regular.

Remark. Theorem 5.5 gives in particular exact regularity of the Bergman projections in the cases where we have already shown that the $\bar{\partial}$-Neumann operators are exactly regular. When Ω is strictly pseudoconvex, Theorem 3.4 implies that for $0 \leq q \leq n$, P_q is exactly regular in Sobolev spaces. More generally, it suffices that Ω is of finite type, compare the discussion after the remarks following the proof of Theorem 3.6. When $1 \leq q \leq n$ and N_q is compact, Theorem 4.6 shows that it is exactly regular; consequently, P_{q-1}, P_q, and P_{q+1} are exactly regular as well.

Proof of Theorem 5.5. Recall from Chapter 2 that

$$P_q = \mathbb{1} - \bar{\partial}^*\bar{\partial}N_q = \bar{\partial}\bar{\partial}^*N_q. \tag{5.30}$$

Next, consider the inner product (f, g) between two $(0, q)$-forms, with g $\bar{\partial}$-closed. Then $(P_{q-1}f, g) = (f, g)$, and if we denote the weight function $e^{-t|z|^2}$ by $w_t(z)$, this inner product satisfies furthermore

$$\begin{aligned}(P_{q-1}f, g) &= (f, g) = (w_{-t}f, g)_t \\ &= (P_{t,q-1}w_{-t}f, g)_t = \left(P_{q-1}w_tP_{t,q-1}w_{-t}f, g\right),\end{aligned} \tag{5.31}$$

where the subscript t denotes the inner product with weight $w_t(z)$. Consequently, for $q \geq 1$,

$$\begin{aligned}P_{q-1} &= P_{q-1}w_tP_{t,q-1}w_{-t} \\ &= (\mathbb{1} - \bar{\partial}^*N_q\bar{\partial})(w_tP_{t,q-1}w_{-t}) \\ &= w_tP_{t,q-1}w_{-t} - \bar{\partial}^*N_q(\bar{\partial}w_t \wedge P_{t,q-1}w_t).\end{aligned} \tag{5.32}$$

For P_{q+1}, we observe that $P_{q+1} - P_{t,q+1} = P_{q+1}(\mathbb{1} - P_{t,q+1})$. Inserting (5.30) for P_{q+1} on the right-hand side and moving $P_{t,q+1}$ to the right gives

$$\begin{aligned}P_{q+1} &= P_{t,q+1} + \bar{\partial}\bar{\partial}^*N_{q+1}(\mathbb{1} - P_{t,q+1}) \\ &= P_{t,q+1} + \bar{\partial}N_q\bar{\partial}^*(\mathbb{1} - P_{t,q+1}) \\ &= P_{t,q+1} + \bar{\partial}N_q(\bar{\partial}^* - \bar{\partial}_t^*)(\mathbb{1} - P_{t,q+1}).\end{aligned} \tag{5.33}$$

We have used here that $\mathbb{1} - P_{t,q+1} \in \ker(\bar{\partial}_t^*)$ (so in particular, it is in the domain of $\bar{\partial}^*$ so that the commutation of $\bar{\partial}^*N_{q+1} = N_q\bar{\partial}^*$ is justified). Note that $\bar{\partial}^* - \bar{\partial}_t^*$ is an operator of order zero. From the weighted theory we know that given any $k \in \mathbb{N}$, we can choose t big enough so that $P_{t,q-1}$, $P_{t,q}$ and $P_{t,q+1}$ are continuous on $W^k(\Omega)$ (at the respective form levels). Thus we have expressed P_{q-1}, P_q, and P_{q+1} in terms of good weighted operators, and operators that are 'as good as N_q' (in view of Corollary 3.3), namely the operators $\bar{\partial}\bar{\partial}^*N_q$, $\bar{\partial}^*N_q$, and $\bar{\partial}N_q$.

We now express N_q in terms of P_{q-1}, P_q, P_{q+1}, and good weighted operators. We have

$$N_q = N_q(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})N_q = (\bar{\partial}^*N_q)^*(\bar{\partial}^*N_q) + (\bar{\partial}^*N_{q+1})(\bar{\partial}^*N_{q+1})^* \tag{5.34}$$

(this observation comes from [125], p. 55, and from [258]). Observe that $\bar{\partial}^*N_q = \bar{\partial}^*N_qP_q$; this is clear on $\ker(\bar{\partial})$, and on $\ker(\bar{\partial})^\perp$, both operators are zero. Thus

$$\begin{aligned}\bar{\partial}^*N_q &= \bar{\partial}^*N_qP_q \\ &= (\mathbb{1} - P_{q-1})\bar{\partial}_t^*N_{t,q}P_q.\end{aligned} \tag{5.35}$$

(5.35) results from interpreting $\bar{\partial}^* N_q$ as the canonical solution operator to $\bar{\partial}$: we may take the solution operator provided by $\bar{\partial}_t^* N_{t,q}$ and then project onto $\ker(\bar{\partial})^\perp$ by $(\mathbb{1} - P_{q-1})$. Since the adjoint in the unweighted metric of $\bar{\partial}_t^* N_{t,q}$ is $w_t(N_{t,q}\bar{\partial})w_{-t}$, it follows from (5.34) that

$$\begin{aligned} N_q &= P_q w_t (N_{t,q}\bar{\partial})\big(w_{-t}(\mathbb{1} - P_{q-1})\bar{\partial}_t^* N_{t,q} P_q\big) \\ &\quad + (\mathbb{1} - P_q)\bar{\partial}_t^* N_{t,q+1} P_{q+1} w_t (N_{t,q+1}\bar{\partial})\big(w_{-t}(\mathbb{1} - P_q)\big) \\ &= P_q w_t N_{t,q}\big(w_{-t} P_q + \bar{\partial} w_{-t} \wedge (\mathbb{1} - P_{q-1})\bar{\partial}_t^* N_{t,q} P_q\big) \\ &\quad + (\mathbb{1} - P_q)\bar{\partial}_t^* N_{t,q+1} P_{q+1} w_t \bar{\partial} N_{t,q}\big(w_{-t}(\mathbb{1} - P_q)\big). \end{aligned} \tag{5.36}$$

To obtain the last equality in (5.36) we have used that $\bar{\partial}(\mathbb{1} - P_{q-1})\bar{\partial}_t^* N_{t,q} P_q = \bar{\partial}\bar{\partial}_t^* N_{t,q} P_q = P_q$ (in particular, $\omega_{-t}(\mathbb{1} - P_{q-1})\bar{\partial}_t^* N_{t,q} P_q$ maps into $\mathrm{dom}(\bar{\partial})$, so that 'breaking up' $(N_{t,q}\bar{\partial})$ is justified, see Remark (ii) after the statement of Theorem 2.9) and that $(N_{t,q+1}\bar{\partial}) = \bar{\partial} N_{t,q}$ (see again Chapter 2).

Combining (5.30), (5.32) (5.33) and (5.36) with Corollary 3.3 (plus interpolation) and Theorem 5.1 proves the theorem. Assume N_q is exactly regular. Then so are $\bar{\partial} N_q$, $\bar{\partial}^* N_q$, and $\bar{\partial}\bar{\partial}^* N_q$, by Corollary 3.3. Fix $s > 0$, and choose t big enough so that all the weighted operators in (5.32) and (5.33) are continuous in W^s-norms. Then (5.30), (5.32), and (5.33) show that P_q, P_{q-1}, and P_{q+1}, respectively, are continuous in W^s-norms; in (5.33), we use that $\bar{\partial}^* - \bar{\partial}_t^*$ is an operator of order zero (i.e., it does not involve derivatives). As $s > 0$ was arbitrary, P_q, P_{q-1}, and P_{q+1} are exactly regular. Conversely, if P_{q-1}, P_q, and P_{q+1} are exactly regular, we use (5.36) in conjunction with Theorem 5.1 to conclude that N_q is exactly regular. The proofs of the assertions about global regularity are similar: one shows that for each positive integer k_1, there exists an integer k_2 such that the operator is continuous from $W^{k_2}(\Omega)$ to $W^{k_1}(\Omega)$. □

Remarks. (i) By writing $\bar{\partial}^* N_q = N_{q-1}\bar{\partial}^*$ in formula (5.34), one can express N_q in terms of N_{q-1} and N_{q+1}.

(ii) N_n is always exactly regular (on a bounded smooth pseudoconvex domain). In fact, more is true: N_n gains two derivatives in Sobolev norms, see (2.95). $P_{n-1} = \mathbb{1} - \bar{\partial}^* N_n \bar{\partial}$ is then (trivially) exactly regular.

(iii) Theorem 5.5 has the following corollary: when N_q and N_{q+3} are exactly (globally) regular, then so are N_{q+1} and N_{q+2}. (That one can bridge a gap of one was observed in [125], p. 55., using Remark (i).) In particular, for dimension $n \le 4$, exact (resp. global) regularity of N_1 implies exact (resp. global) regularity of N_q for all q, in view of Remark (ii). That is, for these low dimensions (essentially $n = 3, 4$), regularity of the $\bar{\partial}$-Neumann operator percolates up the Cauchy–Riemann complex. Recall from Proposition 4.5 and the remark following its proof that compactness and subellipticity percolate up in all dimensions. For regularity, this is open.

(iv) The formulas in the proof of Theorem 5.5 can also be used to obtain pseudolocal estimates for the Bergman projections from those for the $\bar{\partial}$-Neumann problem. We only illustrate the idea and show how to obtain a weak such estimate: the global term is not optimal. But the estimate is sufficient for what we need later. Assume for example that $\bar{\partial}^* N_q$ satisfies pseudolocal estimates near all points of an open subset U of the boundary, say of the form

$$\|\varphi_1 \bar{\partial}^* N_q u\|_s \lesssim \|\varphi_2 u\|_s + \|u\|, \tag{5.37}$$

whenever φ_1 and φ_2 are two smooth cutoff functions with $\varphi_2 \equiv 1$ near the support of φ_1, and whose supports intersect the boundary in U. Then the Bergman projection P_{q-1} satisfies the pseudolocal estimates

$$\|\varphi_1 P_{q-1} u\|_s \lesssim \|\varphi_2 u\|_s + \|u\|_1, \tag{5.38}$$

where $1 \leq q \leq n+1$, with φ_1 and φ_2 as above. (Of course, the case of interest is $s \geq 1$.) First note that $\|\varphi_1 P_{q-1} u\|_s \leq \|\varphi_1 P_{q-1} \varphi_2 u\|_s + \|\varphi_1 P_{q-1}(1-\varphi_2)u\|_s$. Expressing P_{q-1} in the first term by (5.32) and in the second term by the usual formula (Kohn's formula) gives

$$\begin{aligned} \|\varphi_1 P_{q-1} u\|_s \lesssim\ & \|\varphi_1 w_t P_{t,q-1} w_{-t} \varphi_2 u\|_s + \|\varphi_1 \bar{\partial}^* N_q (\bar{\partial} w_t \wedge P_{t,q-1} w_t \varphi_2 u)\|_s \\ & + \|\varphi_1 (1-\varphi_2) u\|_s + \|\varphi_1 \bar{\partial}^* N_q \bar{\partial}(1-\varphi_2) u\|_s, \end{aligned} \tag{5.39}$$

where t is chosen big enough so that $P_{t,q-1}$ is continuous on $W^s_{(0,q)}(\Omega)$. The supports of φ_1 and $(1-\varphi_2)$ are disjoint. This has two consequences. First, the third term on the right-hand side of (5.39) is zero. Second, the pseudolocal estimates for $\bar{\partial}^* N_q$ give in this situation

$$\|\varphi_1 \bar{\partial}^* N_q \bar{\partial}(1-\varphi_2) u\|_s \lesssim \|\bar{\partial}(1-\varphi_2) u\| \lesssim \|u\|_1; \tag{5.40}$$

the local term (obtained when introducing a suitable third cutoff function φ_3 so that φ_1, φ_3, and φ_2 are nested) vanishes as well. The first term on the right of (5.39) is bounded by $\|\varphi_2 u\|_s$, by the Sobolev estimates for $P_{t,q-1}$. To the second term, we also apply the pseudolocal estimates for $\bar{\partial}^* N_q$ and obtain that it is dominated by

$$\begin{aligned} & \|\varphi_2 (\bar{\partial} w_t \wedge P_{t,q-1} w_t \varphi_2 u)\|_s + \|\bar{\partial} w_t \wedge P_{t,q-1} w_t \varphi_2 u\| \\ & \lesssim \|\bar{\partial} w_t \wedge P_{t,q-1} w_t \varphi_2 u\|_s \lesssim \|\varphi_2 u\|_s; \end{aligned} \tag{5.41}$$

the last step again uses the Sobolev estimates for $P_{t,q-1}$. Thus the right-hand side of (5.39) is indeed dominated by the right-hand side of (5.38), and (5.38) is established.

5.4 Benign derivatives

We now return to Sobolev estimates for the unweighted $\bar{\partial}$-Neumann operator. A first general observation is that derivatives of type $(0,1)$ and complex tangential derivatives of type $(1,0)$ are benign for the $\bar{\partial}$-Neumann problem ([48]):

Lemma 5.6. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with defining function ρ, $k \in \mathbb{N}$, $0 \leq q \leq n$, and Y a vector field of type $(1,0)$ with coefficients in $C^\infty(\overline{\Omega})$ with $Y\rho = 0$ on $b\Omega$. Then there is a constant C such that for $u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar\partial^*)$, we have the estimates*

$$\sum_{j,J} \left\| \frac{\partial u_J}{\partial \bar z_j} \right\|^2_{k-1} \leq C(\|\bar\partial u\|^2_{k-1} + \|\bar\partial^* u\|^2_{k-1} + \|u\|^2_{k-1}) \tag{5.42}$$

and

$$\|Yu\|^2_{k-1} \leq C(\|\bar\partial u\|^2_{k-1} + \|\bar\partial^* u\|^2_{k-1} + \|u\|_{k-1}\|u\|_k). \tag{5.43}$$

When $q = 0$, we understand $\bar\partial^*$ to be the zero operator. As an illustration of 'benign for the $\bar\partial$-Neumann problem', consider the following. Suppose we want to estimate $\|N_q u\|_1$. Let X be a field of type $(1,0)$ that is transversal to the boundary ($X\rho \neq 0$ on $b\Omega$). Any derivative D can be written as a sum $aX + Y + \overline{Z}$, where Y and Z are also fields of type $(1,0)$, and Y is tangential at the boundary. Then $\|DN_q u\| \lesssim \|XN_q u\| + \|YN_q u\| + \|\overline{Z}N_q u\|$, and we obtain

$$\begin{aligned} \|N_q u\|_1^2 &\lesssim \|XN_q u\|^2 + \|\bar\partial N_q u\|^2 + \|\bar\partial^* N_q u\|^2 + \|N_q u\|\|N_q u\|_1 \\ &\lesssim \|XN_q u\|^2 + \|u\|^2 + \text{s.c.}\|N_q u\|_1^2 + \text{l.c.}\|N_q u\|^2; \end{aligned} \tag{5.44}$$

at the a priori level (i.e., assuming that $N_q u$ actually is in $W^1_{(0,q)}(\Omega)$), we can absorb the term $\|N_q u\|_1^2$ to get

$$\|N_q u\|_1^2 \lesssim \|XN_q u\|^2 + \|u\|^2. \tag{5.45}$$

That is, in order to prove a Sobolev-1 estimate (at the a priori level), we only have to control a transverse derivative of type $(1,0)$. Similarly, to control the commutators in Theorem 5.7 below, it suffices to have control over the normal $(1,0)$-components.

Proof of Lemma 5.6. When $q = 0$, (5.42) is trivial: $\partial u/\partial \bar z_j$ is a component of $\bar\partial u$. When $q \geq 1$, the case $k = 1$ of (5.42) follows from Proposition 2.4 (with $a \equiv 1$, $\varphi \equiv 0$). To obtain the case $k = 1$ of (5.43) from (5.42), we integrate by parts as follows:

$$\begin{aligned} \|Yu\|^2 &= {\sum_J}' \int_\Omega Yu_J \cdot \overline{Y}\overline{u_J} = -{\sum_J}' \int_\Omega \overline{Y}Yu_J\overline{u_J} + O(\|u\|\|u\|_1) \\ &= -{\sum_J}' \int_\Omega Y\overline{Y}u_J\overline{u_J} - {\sum_J}' \int_\Omega [\overline{Y}, Y]u_J\overline{u_J} + O(\|u\|\|u\|_1) \\ &= {\sum_J}' \int_\Omega \overline{Y}u_J \cdot \overline{\overline{Y}u_J} - {\sum_J}' \int_\Omega [\overline{Y}, Y]u_J\overline{u_J} + O(\|u\|\|u\|_1). \end{aligned} \tag{5.46}$$

Now $\|\bar{Y}u\|^2$ is estimated by (5.42) ($k = 1$), and since the commutator $[\bar{Y}, Y]$ is an operator of order 1, the last two terms in (5.46) are dominated by $\|u\|_1\|u\|$.

To lift the $k = 1$ case of (5.42) to higher norms when $q \geq 1$, we use induction on k. Denote by D^k a derivative of order k. Lemma 2.2 gives

$$\left\|D^k \frac{\partial u}{\partial \bar{z}_j}\right\|^2 \lesssim \left\|D^{k-1} T \frac{\partial u}{\partial \bar{z}_j}\right\|^2 + \left\|D^{k-1} \bar{\partial} \frac{\partial u}{\partial \bar{z}_j}\right\|^2 + \left\|D^{k-1} \theta \frac{\partial u}{\partial \bar{z}_j}\right\|^2 + \left\|D^{k-1} \frac{\partial u}{\partial \bar{z}_j}\right\|^2, \tag{5.47}$$

where T is a tangential derivative (vector field). Commuting $\partial/\partial\bar{z}_j$ with T, $\bar{\partial}$, and θ, respectively gives that the right-hand side of (5.47) is dominated by

$$\left\|\frac{\partial}{\partial \bar{z}_j} T u\right\|_{k-1}^2 + \|\bar{\partial} u\|_k^2 + \|\bar{\partial}^* u\|_k^2 + \|u\|_k^2. \tag{5.48}$$

Letting T act in special boundary charts changes Tu only by 0-th order terms, hence the error we make in the first term in (5.48) is $O(\|u\|_k)$, and so is acceptable. If T acts in special boundary charts, $Tu \in \mathrm{dom}(\bar{\partial}^*)$, so that we an apply the induction assumption to obtain

$$\begin{aligned}\left\|\frac{\partial}{\partial \bar{z}_j} T u\right\|_{k-1}^2 &\lesssim \|\bar{\partial} T u\|_{k-1}^2 + \|\bar{\partial}^* T u\|_{k-1}^2 + \|T u\|_{k-1}^2 \\ &\lesssim \|\bar{\partial} u\|_k^2 + \|\bar{\partial}^* u\|_k^2 + \|u\|_k^2.\end{aligned} \tag{5.49}$$

Combining (5.47)–(5.49) gives the required estimate.

(5.43) follows from (5.42) by integration by parts:

$$\begin{aligned}\|D^{k-1} Y u\|^2 &\lesssim \|Y D^{k-1} u\|^2 + \|u\|_{k-1}^2 \\ &\lesssim \|\bar{Y} D^{k-1} u\|^2 + \|D^k u\| \|D^{k-1} u\| \\ &\lesssim \|D^{k-1} \bar{Y} u\|^2 + \|u\|_{k-1}^2 + \|u\|_k \|u\|_{k-1}.\end{aligned} \tag{5.50}$$

The second inequality in (5.50) is just the above integration by parts argument (5.46) applied to $YD^{k-1}u$ (note that in this argument, it is not used that the form is in the domain of $\bar{\partial}^*$). It now suffices to apply (5.42) to the first term on the right-hand side of (5.50). This completes the proof of Lemma 5.6. □

Remarks. (i) We will often use Lemma 5.6 to estimate the $\mathcal{L}^2$-norm of say, k, derivatives of a from u, one of which is of type $(0, 1)$ or of type $(1, 0)$ and complex tangential. The squares of these $\mathcal{L}^2$-norms are dominated by the right-hand side of (5.42) and (5.43), respectively. This is an immediate consequence of the lemma as formulated. One can commute the type $(0, 1)$ derivative or the complex tangential type $(1, 0)$ derivative to the right, so that they act first and the lemma applies. The error produced by the commutator is $O(\|u\|_{k-1}^2)$ (and so is dominated by the respective right-hand sides).

(ii) We will also use Lemma 5.6 to estimate (suitable) derivatives of the Bergman projection $P_{q-1}u$ of a form u. Again, the lemma as stated does not apply directly, as $P_{q-1}u$ need not be in the domain of $\bar{\partial}^*$. However, $u - P_{q-1}u = \bar{\partial}^* N_q \bar{\partial} u$ is (in fact, it is in $\ker(\bar{\partial}^*)$). The application will be to Sobolev estimates for $P_{q-1}u$, so the error introduced by u is harmless.

5.5 Good vector fields imply Sobolev estimates

Recall how Sobolev estimates were proved in the strictly pseudoconvex case (Theorem 3.4) and in the case where the $\bar{\partial}$-Neumann problem satisfies a compactness estimate (Theorem 4.6). What has to be controlled are error terms arising from commutators between derivatives and $\bar{\partial}$ and $\bar{\partial}^*$. In the strictly pseudoconvex case, the error terms are of order lower than the terms to be estimated, while in the compact case, they are of the same order, but come with a small constant (from the compactness estimate). In both cases, they can be absorbed. In this chapter, the commutators themselves will be small (modulo terms which in view of Lemma 5.6 are under control, see in particular condition (ii) in Theorem 5.7 below).

Remark. There is an observation of Kohn and his school that also relies on having a small factor in front of the commutators. If Ω is a bounded smooth pseudoconvex domain in $\mathbb{C}^n$, then there exists $s_0 > 0$ such that the $\bar{\partial}$-Neumann operators N_q, $1 \leq q \leq n$, are continuous in $W^s_{(0,q)}(\Omega)$ for $0 \leq s \leq s_0$. The reason is that if $\bar{\partial}$ and $\bar{\partial}^*$ are commuted with a (pseudo)differential operator of order s (see (3.67)), the commutator (which is of order s) comes with a factor s in front. Thus when s is small enough, these terms can be absorbed. (For more recent work on how to make the range where estimates hold precise in terms of the Diederich–Fornæss exponent of the domain, see Remark (iii) after Corollary 5.11.)

The following theorem from [48], [51] gives a general sufficient condition for global regularity. Historically, results on Reinhardt domains and other domains with suitable groups of symmetries (genuine [28], [8], [281], [78], [44], [79] or approximate [10]) actually came first, and helped prepare the way for Theorem 5.7. We will discuss these domains in the next section (Section 5.6).

The theorem uses the notion of finite type, which we have discussed only very briefly in Chapter 2, where we also mentioned pseudolocal subelliptic estimates. These estimates are required in the proof of Theorem 5.7. We also need the fact that the set of boundary points of infinite type is closed; this follows from D'Angelo's theorem on the local boundedness of the type (which implies that the set of points of finite type is open; see [89], Theorem 4.11, [92], Theorem 6 in Chapter 4). Alternatively, the reader may change 'the set of boundary points of Ω of infinite type' in Theorem 5.7 to 'the set of weakly pseudoconvex boundary points of Ω'. (In the applications of Theorem 5.7, it will be obvious how to make the analogous modification.) Then only the $1/2$ pseudolocal subelliptic estimates of Theorem 3.6 are required. In important cases, the required vector fields exist on the whole boundary from the outset; in this case, no pseudolocal estimates need to be invoked.

Denote by K the set of boundary points of infinite type.

Theorem 5.7. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function ρ. Suppose there is a positive constant C and open sets $U_1, \dots, U_M$ whose union covers K such that the following holds. For every positive ε there are a neighborhood (in $\mathbb{C}^n$) U_ε of K and vector fields $X_{k,\varepsilon}$, $1 \le k \le M$, of type $(1,0)$, whose coefficients are smooth in U_ε, such that for $1 \le k \le M$*

(i) $|\arg X_{k,\varepsilon}\rho| < \varepsilon$ *and* $|X_{k,\varepsilon}\rho| < C$ *on* U_ε, $C^{-1} < |X_{k,\varepsilon}\rho|$ *on* $U_k \cap U_\varepsilon$; *and*

(ii) $\left|\partial\rho([X_{k,\varepsilon}, \partial/\partial\bar{z}_j])\right| < \varepsilon$ *on* U_ε, $1 \le j \le n$.

Then the $\bar{\partial}$-Neumann operators N_q, $1 \le q \le n$, and the Bergman projections P_q, $0 \le q \le n$, are continuous on $W^s_{(0,q)}(\Omega)$ when $s \ge 0$.

(i) says that $(X_{k,\varepsilon} - \overline{X_{k,\varepsilon}})\rho \approx 0$, meaning that $(X_{k,\varepsilon} - \overline{X_{k,\varepsilon}})$ is approximately tangential on $U_\varepsilon \cap b\Omega$. This property will be used via integration by parts. Moreover, $X_{k,\varepsilon}$ is transverse to $b\Omega$ on $U_\varepsilon \cap U_k \cap b\Omega$, uniformly in ε, so that powers of $X_{k,\varepsilon}$ will control the Sobolev norms to be estimated (in view of Lemma 5.6).

(ii) says that the commutator with derivatives of type $(0,1)$ has small normal $(1,0)$-component in U_ε. This condition replaces the one, used in the work cited above concerning domains with suitable symmetries, that the vector fields have holomorphic coefficients, or equivalently, that their commutators with $\partial/\partial\bar{z}_j$ vanish for $1 \le j \le n$, (so that the parameter ε was absent). That only the normal component of the commutators matters results from Lemma 5.6; other components produce terms that are benign.

Admitting finitely many families turns out to be necessary in order to handle situations where there are otherwise appropriate vector fields that satisfy the transversality condition only on a part of the boundary. This occurs for a natural class of domains that includes Reinhardt domains, see Corollary 5.8 below. Note that the fields $X_{k,\varepsilon}$ are still defined globally, i.e., in all of U_ε, not just in $U_\varepsilon \cap U_k$. Indeed, when $X_{k,\varepsilon}$ is only assumed to be defined in $U_\varepsilon \cap U_k$, with (i) and (ii) holding there, the conclusion of Theorem 5.7 may fail. Examples are provided by the worm domains, see Remark 3 following the proof of Theorem 5.21.

Theorem 5.7 applies widely, for example on all convex domains, and more generally, on domains that admit a defining function whose complex Hessian is positive semidefinite at all boundary points. We will discuss these and several other applications after the proof of the theorem.

Remarks. (i) The special form ε in the bound in the commutator condition (ii) is of course not important. Any function of ε that tends to zero as ε tends to zero will do; one can rescale the families $\{X_{k,\varepsilon}\}$. If the first part of (i) and (ii) hold on K, then by continuity they hold in a sufficiently small neighborhood U_ε.

(ii) The reader should note that condition (ii) can also be viewed as a statement about derivatives of X_ε: $(\partial/\partial\bar{z}_j)X_\varepsilon$ (computed componentwise) should have normal component $O(\varepsilon)$ at points in U_ε. Both points of view are useful.

(iii) Work of Derridj ([102], Théorème 2.6) shows that the assumptions of Theorem 5.7 are optimal in the sense that there need not exist a vector field X of type $(1,0)$ in a neighborhood U of the set of boundary points of infinite type such that $X\rho$ is real in U and $\partial\rho([X, \partial/\partial\bar{z}_j])$ vanishes on U, $1 \le j \le n$. This does leave open the possibility that there would exist a field such that these quantities, while not vanishing in a neighborhood of K, vanish at points of K only. In this context, compare also Remark (ii) following the proof of Corollary 5.16.

Proof of Theorem 5.7. We first prove the case $M = 1$, following (for the most part) [51], [48]. By replacing U_ε by $U_1 \cap U_\varepsilon$, we may assume that $X_{1,\varepsilon} = X_\varepsilon$ is defined in U_ε, and that (i) and (ii) hold there. By Theorem 5.5, the assertions for the Bergman projections P_q and the $\bar\partial$-Neumann operators N_q are equivalent. We will prove the assertions for the P_q. As usual, it suffices by interpolation to prove the case $s = k \in \mathbb{N}$.

The Bergman projection P_{n-1} is continuous in $W^k_{(0,n-1)}(\Omega)$. Namely, $P_{n-1} = \mathbb{1} - \bar\partial^* N_n \bar\partial$, and N_n gains two derivatives in Sobolev norms, see (2.95). The proof now uses a downward induction on the degree q. Assume that P_m is continuous on $W^j_{(0,m)}(\Omega)$ for all $j \in \mathbb{N}$ and $q \le m \le n-1$. We are going to show that then P_{q-1} is likewise continuous on $W^k_{(0,q-1)}(\Omega)$, where k is any (henceforth fixed) positive integer. To do so, we first prove a priori estimates, that is, we will show that $\|P_{q-1}u\|_k \le C_k\|u\|_k$ if both u and $P_{q-1}u$ are in $C^\infty_{(0,q-1)}(\overline{\Omega})$.

We start by showing that under these a priori assumptions, $N_q\bar\partial u \in C^\infty_{(0,q)}(\overline{\Omega})$ as well. This will guarantee that all the norms in the computations below are finite, and that there are no issues with the various integrations by parts. In addition, estimate (5.52) below will be needed in the argument. Recall from the proof of Theorem 5.5 that

$$\begin{aligned} N_q\bar\partial &= (\bar\partial^* N_q)^* = \big((\mathbb{1} - P_{q-1})\bar\partial_t^* N_{t,q} P_q\big)^* \\ &= P_q w_t (N_{t,q}\bar\partial) w_{-t} (\mathbb{1} - P_{q-1}). \end{aligned} \tag{5.51}$$

When $j \in \mathbb{N}$ and t is chosen big enough, the Sobolev estimates for $N_{t,q}\bar\partial$ (see Remark (i) following the proof of Theorem 5.1) in conjunction with the induction assumption on P_q show that $N_q\bar\partial u \in W^j_{(0,q)}(\Omega)$, with the estimate

$$\|N_q\bar\partial u\|_j \le C_j(\|u\|_j + \|P_{q-1}u\|_j). \tag{5.52}$$

Thus $N_q\bar\partial u$ belongs to $W^j_{(0,q)}(\Omega)$ for all $j \in \mathbb{N}$, hence to $C^\infty_{(0,q)}(\overline{\Omega})$ (by the Sobolev imbedding theorem).

To prove the desired a priori estimate, fix $\varepsilon > 0$ (its value will be specified later), and denote by X_ε the corresponding vector field from the assumption in the theorem (with $M = 1$). We first show that in order to estimate $\|P_{q-1}u\|_k^2$, it suffices to estimate

$\|\varphi_\varepsilon X_\varepsilon^k P_{q-1}u\|^2$ (φ_ε is a suitable cutoff function). This is where the transversality condition in (i) is needed.

In order to apply Lemma 5.6, we need a form that is in the domain of $\bar{\partial}^*$, and so cannot directly apply this lemma to $P_{q-1}u$. However, $(\mathbb{1} - P_{q-1})u$ is in the domain (in fact, in the kernel) of $\bar{\partial}^*$, and Lemma 5.6 applies. Choose a smooth cutoff function φ_ε that is supported in U_ε and is identically equal to 1 in a neighborhood of the compact set of boundary points of infinite type. Then $\varphi_\varepsilon(u - P_{q-1}u) \in \mathrm{dom}(\bar{\partial}^*)$. On the support of $(1 - \varphi_\varepsilon)$, we have subelliptic pseudolocal estimates ([70], or Theorem 3.6 for the modified version of Theorem 5.7); in particular, (5.37) holds (U is the set of points of finite type, or the set of strictly pseudoconvex boundary points for the modified version; see also Remark (ii) after the proof of Theorem 3.6). Therefore, (5.38) also holds, and we obtain

$$\begin{aligned} \|P_{q-1}u\|_k^2 &\lesssim \|\varphi_\varepsilon P_{q-1}u\|_k^2 + \|(1-\varphi_\varepsilon)P_{q-1}u\|_k^2 \\ &\lesssim \|\varphi_\varepsilon P_{q-1}u\|_k^2 + C_\varepsilon\|u\|_k^2 \\ &\lesssim \|\varphi_\varepsilon(\mathbb{1} - P_{q-1})u\|_k^2 + C_\varepsilon\|u\|_k^2 \end{aligned} \tag{5.53}$$

(note that $k \geq 1$). Observe that on the support of φ_ε, the normal (1,0) derivative is of the form $a_\varepsilon X_\varepsilon + Y_{1,\varepsilon}$, where $Y_{1,\varepsilon}$ is a tangential field of type $(1,0)$ and a_ε is smooth, and bounded independently of ε (from assumption (i) in the theorem). Any derivative of order k of $\varphi_\varepsilon(\mathbb{1} - P_{q-1})u$ can thus be written as a smooth multiple of $(a_\varepsilon X_\varepsilon)^k(\mathbb{1} - P_{q-1})u$ plus a sum of terms, each of which contains at least one derivative that is complex tangential of type $(1,0)$ or that is of type $(0,1)$. Modulo commutators which are of order at most $(k-1)$, we may assume that these latter derivatives act first, so that the $\mathcal{L}^2$-norm of these terms is bounded by the $(k-1)$-norm of the derivatives. Applying Lemma 5.6 to these terms gives

$$\begin{aligned} \|\varphi_\varepsilon(\mathbb{1} - P_{q-1})u\|_k^2 &\lesssim \|(a_\varepsilon X_\varepsilon)^k\varphi_\varepsilon(\mathbb{1} - P_{q-1})u\|^2 + C_\varepsilon\{\|\bar{\partial}(\varphi_\varepsilon(\mathbb{1} - P_{q-1})u)\|_{k-1}^2 \\ &\qquad + \|\bar{\partial}^*(\varphi_\varepsilon(\mathbb{1} - P_{q-1})u)\|_{k-1}^2 + \|\varphi_\varepsilon(\mathbb{1} - P_{q-1})u\|_{k-1}^2 \\ &\qquad + \|\varphi_\varepsilon(\mathbb{1} - P_{q-1})u\|_{k-1}\|\varphi_\varepsilon(\mathbb{1} - P_{q-1})u\|_k\} \\ &\lesssim \|X_\varepsilon^k\varphi_\varepsilon P_{q-1}u\|^2 + C_\varepsilon\|u\|_k^2 \\ &\qquad + C_\varepsilon\|P_{q-1}u\|_{k-1}^2 + \text{s.c.}\|\varphi_\varepsilon(\mathbb{1} - P_{q-1})u\|_k^2, \end{aligned} \tag{5.54}$$

where the constant in $\lesssim$ does not depend on ε, and where s.c. denotes a small constant (as small as we want, at the expense of C_ε). We have also used that if a derivative hits a_ε, there will be at most $(k-1)$ derivatives on $(\mathbb{1}-P_{q-1})u$, that a_ε is bounded independently of ε, that commutators between φ_ε and $\bar{\partial}$ and $\bar{\partial}^*$, respectively, are operators of order zero, and the inequality $ab \leq \text{l.c.}a^2 + \text{s.c.}b^2$ $(a, b \geq 0)$. After absorbing the last term in (5.54) and inserting the resulting estimate into (5.53), we have

$$\|P_{q-1}u\|_k^2 \leq C\|X_\varepsilon^k\varphi_\varepsilon P_{q-1}u\|^2 + C_\varepsilon(\|u\|_k^2 + \|P_{q-1}u\|_{k-1}^2); \tag{5.55}$$

with the constant C independent of ε.

The main term in (5.55) is $\|X_\varepsilon^k \varphi_\varepsilon P_{q-1}u\|^2 = \big(X_\varepsilon^k \varphi_\varepsilon P_{q-1}u, X_\varepsilon^k \varphi_\varepsilon P_{q-1}u\big)$. We want to integrate by parts all the factors X_ε to the right-hand side of the inner product (it will be clear below why), but at the same time, we would like to avoid boundary terms as much as possible. If $X_\varepsilon\rho$ were real, the standard way to achieve this is to consider $X_\varepsilon - \overline{X_\varepsilon}$, which is tangential. The error this introduces consists of terms that contain at least one $(0,1)$-derivative; therefore, it is benign. In our case, $X_\varepsilon\rho$ is only approximately real, and we modify $X_\varepsilon - \overline{X_\varepsilon}$ in such a way that the error is small: there is a smooth real valued function b_ε of absolute value not exceeding $2C\varepsilon$ such that $X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i L_n$ is tangential, where C is the constant from (i) in Theorem 5.7, and $L_n = (1/|\partial\rho|^2)\sum_{j=1}^n (\partial\rho/\partial\bar{z}_j)\partial/\partial z_j$ (so that $L_n\rho = 1$). Namely, set $b_\varepsilon = i(X_\varepsilon - \overline{X_\varepsilon})\rho$. Then

$$
\begin{aligned}
&\|X_\varepsilon^k \varphi_\varepsilon P_{q-1}u\|^2 \\
&\quad \le \Big|\big((X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i L_n)^k \varphi_\varepsilon P_{q-1}u, (X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i \overline{L_n})^k \varphi_\varepsilon P_{q-1}u\big)\Big| \\
&\qquad + C_\varepsilon \|P_{q-1}u\|_k (\|P_{q-1}u\|_{k-1} + \|u\|_k) + C\varepsilon\|P_{q-1}u\|_k^2,
\end{aligned}
\tag{5.56}
$$

where the constant C in the last term is independent of ε (but is not necessarily the constant from (i)). (5.56) arises from arguments analogous to those that led to (5.54), as follows. In the main term on the right-hand side, consider first terms that contain at least one derivative of type $(0,1)$. Modulo commutators that are of order at most $k-1$, we can again assume that this derivative acts first. Consequently, these terms can be estimated using (5.43) in Lemma 5.6 (applied to $u - P_{q-1}u$); they are dominated by $C_\varepsilon\|P_{q-1}u\|_k(\|P_{q-1}u\|_{k-1} + \|u\|_k)$. The remaining terms all have $X_\varepsilon^k P_{q-1}u$ on the right-hand side of the inner product. Among these, the contribution from the term that has k factors X_ε on the left-hand side of the inner product gives the left-hand side of (5.56). In the remaining terms, there are $m \ge 1$ factors $b_\varepsilon i L_n$. Modulo commutators that are benign, we can assume that these factors act last, so that we have $(b_\varepsilon i L_n)^m (X_\varepsilon)^{k-m} P_{q-1}u$. Note that $X_\varepsilon = c_\varepsilon L_n + Y_\varepsilon$, where the function c_ε is smooth and bounded (on the support of φ_ε) uniformly in ε, and $Y_\varepsilon\rho = 0$. Therefore, the inner product of these last terms with $X_\varepsilon^k P_{q-1}u$ has the form

$$
\big((b_\varepsilon i L_n)^m (c_\varepsilon L_n + Y_\varepsilon)^{k-m} P_{q-1}u, (c_\varepsilon L_n + Y_\varepsilon)^k P_{q-1}u\big). \tag{5.57}
$$

Terms containing a factor Y_ε are estimated via (5.43) in Lemma 5.6. This implies commuting derivatives so that Y_ε acts first and replacing $P_{q-1}u$ by $u - P_{q-1}u$, which produces an error that is dominated by $C_\varepsilon\|P_{q-1}u\|_k(\|u\|_k + \|P_{q-1}u\|_{k-1})$; this error is thus dominated by the right-hand side of (5.56). For the remaining terms, note that $(b_\varepsilon i L_n)^m (c_\varepsilon L_n)^{k-m} = (b_\varepsilon i)^m c_\varepsilon^{k-m} L_n^k$ plus lower order terms. Consequently the inner product with $(c_\varepsilon L_n)^k P_{q-1}u$ is dominated by the last two terms in (5.56) (because of the uniform bounds on b_ε and c_ε). This proves (5.56).

In the main term in (5.56) integrate by parts k times to bound it from above by

$$
\begin{aligned}
&\big|\big(P_{q-1}u, \varphi_\varepsilon(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i \overline{L_n})^{2k} \varphi_\varepsilon P_{q-1}u\big)\big| + C_\varepsilon\|P_{q-1}u\|_{k-1}\|P_{q-1}u\|_k \\
&\quad = \big|\big(u - \bar{\partial}^* N_q \bar{\partial} u, \varphi_\varepsilon(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i \overline{L_n})^{2k} \varphi_\varepsilon P_{q-1}u\big)\big| \\
&\qquad + C_\varepsilon\|P_{q-1}u\|_{k-1}\|P_{q-1}u\|_k.
\end{aligned}
\tag{5.58}
$$

Note that in the error terms, one can move tangential derivatives from one side of the inner product to the other, as needed, before applying the Cauchy–Schwarz inequality. In the main term in (5.58), the contribution coming from u, again by suitable integrations by part, is of order $C_\varepsilon\|u\|_k\|P_{q-1}u\|_k \le \text{s.c.}\|P_{q-1}u\|_k^2 + \text{l.c.}\|u\|_k^2$, with s.c. independent of ε, and so is benign for the estimate we seek to establish: the term s.c.$\|P_{q-1}u\|_k^2$ can be absorbed. In the contribution from $\bar\partial^* N_q\bar\partial u$, we move $\bar\partial^*$ to the right-hand side of the inner product as $\bar\partial$. This is the key step as far as the role of the commutator condition (ii) in the theorem is concerned. Because $\bar\partial P_{q-1}u = 0$, we only get terms where $\bar\partial$ acts on one of the factors φ_ε or on $(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{2k}$. We can dominate the former by $C_\varepsilon\|u\|_k$, via the pseudolocal subelliptic estimates on the support of φ_ε, as in (5.53) (again integrating tangential derivatives by parts to the left, as needed). In the latter, we commute $\bar\partial$ with $(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{2k}$ to obtain a term where $\bar\partial$ acts again on φ_ε (so that the term is dominated by $\|u\|_k$). The resulting commutator to be estimated is

$$\begin{aligned}
&\big[\bar\partial, (X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{2k}\big] \\
&\qquad = 2k(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{2k-1}\big[\bar\partial, X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n}\big] \\
&\qquad\quad + \text{terms of the form } (X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{2k-s} \\
&\qquad\quad \times \big[\underbrace{[\dots[\bar\partial, X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n}], X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n}]\dots, X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n}}_{s\text{-fold iterated commutator}}\big],
\end{aligned} \tag{5.59}$$

with $2 \le s \le 2k$, see (3.54). Note that the iterated commutators in (5.59) are first order operators, so integrating up to $k-s+1 \le k-1$ factors of $(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})$ by parts shows that the contribution from these terms is bounded by $C_\varepsilon\|N_q\bar\partial u\|_{k-1}\|P_{q-1}u\|_k$. Putting these estimates together gives

$$\begin{aligned}
&\big|\big(\bar\partial^* N_q\bar\partial u, \varphi_\varepsilon(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{2k}\varphi_\varepsilon P_{q-1}u\big)\big| \\
&\quad \lesssim \big|\big(N_q\bar\partial u, \varphi_\varepsilon(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{2k-1}[\bar\partial, X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n}]\varphi_\varepsilon P_{q-1}u\big)\big| \\
&\qquad + C_\varepsilon\big(\|u\|_k^2 + \|N_q\bar\partial u\|_{k-1}\|P_{q-1}u\|_k\big) \\
&\quad \lesssim \big|\big((X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon iL_n)^k N_q\bar\partial u, \dots \\
&\qquad \dots\varphi_\varepsilon(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n})^{k-1}[\bar\partial, X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i\overline{L_n}]\varphi_\varepsilon P_{q-1}u\big)\big| \\
&\qquad + C_\varepsilon(\|u\|_k^2 + \|N_q\bar\partial u\|_{k-1}\|P_{q-1}u\|_k).
\end{aligned} \tag{5.60}$$

In the main term in (5.60), we use assumption (ii) of the theorem: on the support of φ_ε, $[\bar\partial, X_\varepsilon - \overline{X_\varepsilon} + bi\overline{L_n}]$ can be written as a sum of derivatives of type $(0, 1)$, of tangential derivatives of type $(1, 0)$, and a function of modulus less than ε times L_n. Terms of the first two types are again estimated using Lemma 5.6 (commuting derivatives and replacing $P_{q-1}u$ by $u - P_{q-1}u$). So we obtain an upper bound for the main term in

(5.60) of the form

$$\begin{aligned} &\big\|(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i L_n)^k N_q \bar{\partial} u\big\| \times \cdots \\ &\qquad \cdots \times \big(C_\varepsilon(\|u\|_k + \|P_{q-1}u\|_k^{1/2}\|P_{q-1}u\|_{k-1}^{1/2}) + \varepsilon\|P_{q-1}u\|_k\big) \\ &\qquad \lesssim \big\|(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i L_n)^k N_q \bar{\partial} u\big\| \big(C_\varepsilon(\|u\|_k + \|P_{q-1}u\|_{k-1}) + \varepsilon\|P_{q-1}u\|_k\big). \end{aligned} \tag{5.61}$$

Write $X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i L_n$ as $(1/a_\varepsilon + b_\varepsilon i)$ times L_n plus a derivative of type $(0,1)$ plus a tangential derivative of type $(1,0)$, and apply Lemma 5.6 once more. Note that the k-norm of $N_q\bar{\partial}u$ only arises when none of the derivatives act on $(1/a_\varepsilon + b_\varepsilon i)$; this allows the constant in front of $\|N_q\bar{\partial}u\|_k^2$ in the following estimate to be chosen independently of ε (in view of (i) in the theorem and the fact that $b_\varepsilon = O(\varepsilon)$):

$$\begin{aligned} &\big\|(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i L_n)^k N_q \bar{\partial} u\big\|^2 \\ &\qquad \le C\|N_q\bar{\partial}u\|_k^2 + C_\varepsilon\big(\|\bar{\partial}^* N_q\bar{\partial}u\|_{k-1}^2 + \|N_q\bar{\partial}u\|_{k-1}\|N_q\bar{\partial}u\|_k\big) \\ &\qquad \le C\|N_q\bar{\partial}u\|_k^2 + C_\varepsilon\big(\|P_{q-1}u\|_{k-1}^2 + \|u\|_{k-1}^2\big). \end{aligned} \tag{5.62}$$

In the last inequality in (5.62), we have expressed $\bar{\partial}^* N_q\bar{\partial}u$ as $u - P_{q-1}u$, and we have used the interpolation inequality $\|N_q\bar{\partial}u\|_{k-1} \le \text{s.c.}\|N_q\bar{\partial}u\|_k + \text{l.c.}\|N_q\bar{\partial}u\|$. (Also note that $\|N_q\bar{\partial}u\|^2 \lesssim \|u\|^2 \le \|u\|_{k-1}^2$.) Combining estimates (5.55)–(5.62), using again the inequality $ab \le \text{s.c.}a^2 + \text{l.c.}b^2$ $(a, b \ge 0)$, as well as the interpolation inequality $\|u\|_{k-1} \le \text{s.c.}\|u\|_k + \text{l.c.}\|u\|$, and absorbing terms, we obtain the estimate

$$\|P_{q-1}u\|_k^2 \le \varepsilon C\left(\|P_{q-1}u\|_k^2 + \|N_q\bar{\partial}u\|_k^2\right) + C_\varepsilon\|u\|_k^2, \tag{5.63}$$

with C independent of ε. Finally, inserting (5.52) into (5.63) and choosing ε small enough gives the a priori estimate

$$\|P_{q-1}u\|_k^2 \le C\|u\|_k^2, \quad u \text{ and } P_{q-1}u \in C^\infty_{(0,q)}(\overline{\Omega}). \tag{5.64}$$

In order to turn this a priori estimate into a genuine estimate, we use a method different from elliptic regularization. Fix a number M so large that $\rho_\delta(z) := \rho(z) + \delta e^{M|z|^2}$ defines a smooth strictly pseudoconvex subdomain Ω_δ of Ω for $\delta > 0$ small enough (see Lemma 2.5). Then on Ω_δ, $P_{q-1,\delta}u \in C^\infty_{(0,q-1)}(\overline{\Omega}_\delta)$ if u is, by Theorems 3.4 and 5.5. We modify the induction hypothesis and require that Sobolev estimates hold uniformly in δ on Ω_δ, for δ small enough (k is fixed). The above estimates can be carried out, uniformly in δ, on Ω_δ, by using the fields X_ε from the theorem. Note that the pseudolocal subelliptic estimates (alternatively, the pseudolocal estimates from Theorem 3.6) on Ω_δ on the support of $(1 - \varphi_\varepsilon)$ are also uniform in δ. If $u \in C^\infty_{(0,q)}(\overline{\Omega})$, $P_{q-1,\delta}u \in C^\infty_{(0,q)}(\overline{\Omega}_\delta)$, by Theorem 3.4, and these estimates apply. We obtain

$$\|P_{q-1,\delta}u\|_k \le C_k\|u\|_{k,\Omega_\delta} \le C_k\|u\|_k, \tag{5.65}$$

with C_k independent of δ. A subsequence of the extensions by zero of the $P_{q-1,\delta}u$ converges weakly in $\mathcal{L}^2_{(0,q-1)}(\Omega)$ (since the $P_{q-1,\delta}u$ form a bounded sequence in

$\mathcal{L}^2_{(0,q-1)}(\Omega)$). Pairing with $\bar{\partial}$-closed forms in $\mathcal{L}^2_{(0,q-1)}(\Omega)$ shows that this weak limit is $P_{q-1}u$. If V is a fixed relatively compact subdomain of Ω, then a further subsequence will converge weakly in $W^k_{(0,q-1)}(V)$, by (5.65). But this weak limit must agree with the weak limit in $\mathcal{L}^2_{(0,q)}(V)$ which is (the restriction to V of) $P_{q-1}u$. Consequently, $P_{q-1}u$ has W^k-norm on V dominated by $\|u\|_k$, independently of V. Since V was arbitrary, it follows that $P_{q-1}u \in W^k_{(0,q-1)}(\Omega)$, and that estimate (5.64) holds. Since $C^\infty_{(0,q-1)}(\overline{\Omega})$ is dense in $W^k_{(0,q-1)}(\Omega)$, this estimate carries over to all $u \in W^k_{(0,q-1)}(\Omega)$. This completes the proof of Theorem 5.7 in the case when $M = 1$.

It remains to indicate the necessary modifications in the proof when there are finitely many families $\{X_{j,\varepsilon}\}$, $1 \le j \le M$, as in the theorem. Let $U_1, \dots, U_M$ be the neighborhoods from the theorem, and choose a partition of unity $\{\chi_j\}_{j=1}^M$ of a neighborhood of K such that $\operatorname{supp}(\chi_j) \subseteq U_j$. We only have to estimate $\|\varphi_\varepsilon \chi_j P_{q-1}u\|_k^2$ for $1 \le j \le M$. The analogue of (5.54) becomes

$$\begin{aligned} \|\varphi_\varepsilon \chi_j (\mathbb{1} - P_{q-1})u\|_k^2 \lesssim \|X^k_{j,\varepsilon}\varphi_\varepsilon \chi_j P_{q-1}u\|^2 + C_{j,\varepsilon}\|u\|_k^2 \\ + C_{j,\varepsilon}\|P_{q-1}u\|_{k-1}^2 + \text{s.c.}\|\varphi_\varepsilon \chi_j (\mathbb{1} - P_{q-1})u\|_k^2, \end{aligned} \tag{5.66}$$

where $1 \le j \le M$. Since

$$\begin{aligned} \|X^k_{j,\varepsilon}\varphi_\varepsilon \chi_j P_{q-1}u\|^2 &\lesssim \|\chi_j X^k_{j,\varepsilon}\varphi_\varepsilon P_{q-1}u\|^2 + C_{j,\varepsilon}(\|P_{q-1}u\|_{k-1}^2) \\ &\lesssim \|X^k_{j,\varepsilon}\varphi_\varepsilon P_{q-1}u\|^2 + C_{j,\varepsilon}(\|P_{q-1}u\|_{k-1}^2), \end{aligned} \tag{5.67}$$

we only have to estimate $\|X^k_{j,\varepsilon}\varphi_\varepsilon P_{q-1}u\|^2$ for $1 \le j \le M$. This is done exactly as above. The proof of the a priori estimate is complete.

Finally, this a priori estimate is turned into a genuine estimate as in the case $M = 1$ above. □

5.6 Domains with symmetries

Before embarking on a systematic study of when the families of vector fields needed in Theorem 5.7 exist, we give a first application to the results mentioned earlier as precursors. A domain is a Reinhardt (or multi-circular) domain if whenever $z = (z_1, \dots, z_n) \in \Omega$, then so is $w = (e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)$, for all $(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$. This class of domains plays an important role in the function theory of several complex variables ([177]): for example, the domain of convergence of a power series in several variables is a Reinhardt domain (see e.g. [207], [259], [177]). More generally, we consider domains with transverse symmetries. This notion, introduced in [8], means the following. Consider a Lie group G of automorphisms of the smooth bounded domain Ω such that the map $G \times \Omega \to \Omega$, $(g, z) \to g(z)$ extends to a smooth map of $G \times \overline{\Omega} \to \overline{\Omega}$. We say that G acts transversely on Ω when for each $P \in b\Omega$, the image of the tangent map $(\tau_P)_*\colon T_{g_0}G \to T_P(b\Omega)$ (where $g_0 \in G$ is the identity) induced by the map $\tau_P : G \to b\Omega$, $g \to g(P)$, is *not* contained in $T^{\mathbb{C}}_P(b\Omega)$. Finally, we say that Ω has transverse symmetries if Ω admits a Lie group of symmetries as above that acts transversely on Ω. If Ω is Reinhardt, one takes the n-torus $\mathbb{T}^n$ for G, with the obvious action $(\theta_1, \dots, \theta_n) \to (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$. It is clear that this action extends smoothly to the closure. To see that it is transversal, fix a boundary point $P = (\zeta_1, \dots, \zeta_m, 0, \dots, 0)$, $1 \le m \le n$, and $\zeta_j \ne 0$, $1 \le j \le m$. In other words, P has precisely the first m coordinates nonzero. It is not hard to see that smoothness of Ω excludes the origin from the boundary, so that P has at least one nonzero coordinate. For simplicity, we assume that all nonzero coordinates come first. P is fixed under rotations in the last $(n-m)$ variables, and rotations take normals to normals (they are unitary). As a result, the complex normal at P also has its last $(n-m)$ coordinates equal to zero. Consequently, the vectors $(0, \dots, 0, w_{m+1}, \dots, w_n)$ belong to $T_P^{\mathbb{C}}(b\Omega)$ for any $(w_{m+1}, \dots, w_n) \in \mathbb{C}^{n-m}$. Consider the action of the m-torus on Ω given by $(\theta_1, \dots, \theta_m) \to R(\theta_1, \dots, \theta_m)$, where $R(\theta_1, \dots, \theta_m)(z) = (e^{i\theta_1} z_1, \dots, e^{i\theta_m} z_m, z_{m+1}, \dots, z_n)$. The associated tangent map at the origin, R_0^*, maps the tangent space to $\mathbb{R}^m$ at 0 into the tangent space to Ω at P. The image is given by the span over $\mathbb{R}$ of the vectors $(iz_1, 0, \dots, 0), \dots, (0, \dots, iz_m, 0, \dots, 0)$. If $R_0^*(\mathbb{R}^m)$ were contained in $T_P^{\mathbb{C}}(b\Omega)$, then $T_P^{\mathbb{C}}(b\Omega)$ would also contain the span over $\mathbb{C}$ of the vectors $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1, 0 \dots, 0)$. With what was said above about $T_P^{\mathbb{C}}(b\Omega)$, this would imply that $T_P^{\mathbb{C}}(b\Omega)$ is all of $\mathbb{C}^n$, a contradiction. We conclude that there exists $\theta_P \in \mathbb{R}^m$ such that $R_0^*(\theta_P)$ is transverse to $T_P^{\mathbb{C}}(b\Omega)$. So Reinhardt domains have transverse symmetries and are covered by Corollary 5.8. Examples where the symmetries are not rotational are obtained from suitable biholomorphic images of Reinhardt domains.

Corollary 5.8. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with transverse symmetries. Then the $\bar{\partial}$-Neumann operator N_q is continuous in $W^s_{(0,q)}(\Omega)$, for $s \ge 0$ and $1 \le q \le n$. The Bergman projection P_q is continuous on $W^s_{(0,q)}(\Omega)$ when $s \ge 0$ and $0 \le q \le n$.*

Remarks. (i) The class of smooth bounded pseudoconvex domains that admit transverse symmetries is invariant under biholomorphisms: if Ω_1 and Ω_2 are biholomorphic, and Ω_1 admits transverse symmetries, then so does Ω_2. It suffices to note that in view of Corollary 5.8 and Bell's regularity theorem ([23]), the biholomorphism extends to a diffeomorphism of the closures.

(ii) In the case of Reinhardt domains, Corollary 5.8 is in [79]. In [281], exact global regularity for the Bergman projection P_0 was shown to hold on any smooth bounded domain with transverse symmetries, pseudoconvex or not.

(iii) For results when the symmetries are only assumed to act transversely on the complement of a compact subset of the boundary that satisfies certain properties (for example property (P_1)), see [78], [44].

(iv) The pullback induced by a rotation on $\mathcal{L}^2_{(0,q)}(\Omega)$ commutes with $\bar{\partial}$; since it is an isometry, it also commutes with $\bar{\partial}^*$, hence with N_q. One can exploit this observation to prove Corollary 5.8 when Ω is a Reinhardt domain; for details, see [281], [44], [79].

Proof of Corollary 5.8. We will construct vector fields as required in Theorem 5.7 that do not depend on ε. Moreover, $X_{k,\varepsilon}\rho = X_k\rho$ will be exactly real on $b\Omega$.

Let $P \in b\Omega$. By assumption, there exists a θ_P in the tangent space to G at the identity $g_0 \in G$ such that $(\tau_P)_*(\theta_P) \notin T_P^{\mathbb{C}}(b\Omega)$. Choose a smooth curve $g(t)$ in G with $g(0) = g_0$ and $(d/dt)g(t)|_{t=0} = \theta_P$. Denote by $X_P(z)$ the $(1,0)$-vector field on $\overline{\Omega}$ given by $X_P(z) = (d/dt)\{g(t)(z)\}|_{t=0}$. X_P is smooth on $\overline{\Omega}$ (because the action of G is smooth on $G \times \overline{\Omega}$), it has holomorphic coefficients (because for t fixed, $g(t)(z)$ is holomorphic in z), and $X_P + \overline{X_P}$ is tangential to $b\Omega$. X_P is defined only on $\overline{\Omega}$, but it can be continued as a field with smooth coefficients into a neighborhood of $\overline{\Omega}$. Then $X_P\rho \neq 0$ in a neighborhood U_P of P (ρ is a defining function for Ω), and (shrinking U_P if necessary), we may assume that $|X_P\rho| \geq c_P > 0$. We can cover the compact set K by finitely many of the U_P's, say $U_{P_1}, \ldots, U_{P_M}$. The families $X_{k,\varepsilon} = iX_{P_k}$ for all $\varepsilon > 0$ satisfy the assumptions in Theorem 5.7, with U_ε chosen small enough. □

5.7 Commutators with strictly pseudoconvex directions

We now begin a systematic study of when families of vector fields as in Theorem 5.7 exist. A key observation is that the commutator conditions in part (ii) of Theorem 5.7 for commutators in strictly pseudoconvex directions as well as in the complex normal direction come for free ([48]). In the context of a nondegenerate Levi form and real analytic boundary, this observation goes back at least to [204] (see Section 3) and [103] (see Propositions 1–5). The observation is crucial, and we formulate the following version of Theorem 5.7, which is implicit in [48], [51]. For clarity, we only formulate the version for one family of vector fields, the extension to finitely many as in Theorem 5.7 being straightforward. $\mathcal{N}(P)$ denotes the null space of the Levi form at a boundary point P; recall that K is the set of boundary points of infinite type.

Theorem 5.9. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function ρ. Suppose there is a positive constant C such that the following holds. For every positive ε there is a vector field X_ε of type $(1,0)$ whose coefficients are smooth in a neighborhood U_ε of K, such that*

(i) *$|\arg X_\varepsilon\rho| < \varepsilon$ and $C^{-1} < |X_\varepsilon\rho| < C$ on K; and*

(ii) *$\left|\partial\rho([X_\varepsilon, \overline{L}])(P)\right| < \varepsilon$ whenever L is a tangential $(1,0)$-field of unit length near $P \in K$ with $L(P) \in \mathcal{N}(P)$.*

Then, the assumptions of Theorem 5.7 are satisfied. Consequently, the $\bar{\partial}$-Neumann operators N_q, $1 \leq q \leq n$, and the Bergman projections P_q, $0 \leq q \leq n$, are continuous on $W^s_{(0,q)}(\Omega)$ when $s \geq 0$.

Note that whether such a family $\{X_\varepsilon\}$ exists does not depend on the choice of the defining function ρ.

Proof of Theorem 5.9. Suppose first that we know that $\partial\rho([X_\varepsilon, \overline{L}])$ is $O(\varepsilon)$ for all tangential unit vector fields L of type $(1,0)$. By shrinking U_ε if necessary, we may assume that both this and (i) hold in U_ε. Then the fields X_ε can be extended from the boundary such that (ii) in Theorem 5.7 holds. If L_n denotes the complex normal, it suffices to prescribe the real normal derivatives $\mathrm{Re}\,L_n(X_\varepsilon)_j$ of the coefficients $(X_\varepsilon)_j$ of X_ε at points of U_ε in such a way that $\overline{L_n}(X_\varepsilon)_j = (\mathrm{Re}\,L_n - i\,\mathrm{Im}\,L_n)(X_\varepsilon)_j = 0$ in U_ε. Note that $\mathrm{Im}\,L_n(X_\varepsilon)_j$ is a tangential derivative. By continuity, we then have $\partial\rho([X_\varepsilon, \overline{L_k}]) = O(\varepsilon)$ for $1 \le k \le n$ in a neighborhood of K in $\mathbb{C}^n$. Similarly, (i) holds for the extended field in a neighborhood of K.

Now denote by $\{X_\varepsilon\}$ the family from the theorem. We modify X_ε to obtain a field Y_ε as follows. Fix $P \in K$. Near P we look for Y_ε in the form $Y_{\varepsilon,P} = X_\varepsilon + \sum_{j=1}^{n-1} a_j^\varepsilon L_j$, where $\{L_1, \dots, L_{n-1}\}$ is a basis for $T^{(1,0)}(b\Omega)$ near P, and a_j^ε are constants. We have

$$\partial\rho([Y_{\varepsilon,P}, \overline{L_k}]) = \partial\rho([X_\varepsilon, \overline{L_k}]) + \sum_{j=1}^{n-1} a_j^\varepsilon \partial\rho([L_j, \overline{L_k}]), \quad 1 \le k \le n-1. \tag{5.68}$$

We may assume that the basis $\{L_1, \dots, L_{n-1}\}$ is chosen so that it diagonalizes the Levi form (see (2.45)) $\left(\partial\rho([L_j, \overline{L_k}])\right)_{j,k=1}^{n-1}$ at P and so that $\{L_1(P), \dots, L_m(P)\}$ span $\mathcal{N}(P)$ $(1 \le m \le n-1)$. (5.68) shows that if we set

$$a_j^\varepsilon = 0, \quad 1 \le j \le m, \tag{5.69}$$

and

$$a_j^\varepsilon = -\frac{\partial\rho([X_\varepsilon, \overline{L_j}])(P)}{\partial\rho([L_j, \overline{L_j}])(P)}, \quad m+1 \le j \le n-1, \tag{5.70}$$

then the field $Y_{\varepsilon,P} = X_\varepsilon + \sum_{j=1}^{n-1} a_j^\varepsilon L_j$, defined near P, satisfies

$$\partial\rho\left([Y_{\varepsilon,P}, \overline{L_k}]\right)(P) = O(\varepsilon), \quad 1 \le k \le n-1. \tag{5.71}$$

By continuity, this holds in a neighborhood V_ε of P. Moreover, $\partial\rho(Y_{\varepsilon,P}) = \partial\rho(X_\varepsilon)$. Via a partition of unity $\{\chi_s\}_{s=1}^M$ of a neighborhood U_ε of K, subordinate to a finite cover of the compact set K by V_ε's, we obtain a field $Y_\varepsilon = \sum_{s=1}^M \chi_s Y_{\varepsilon,P_s}$. In U_ε, this field satisfies, for any complex tangential $(1,0)$-field L of unit length:

$$\begin{aligned}
\partial\rho\left([Y_\varepsilon, \overline{L}]\right) &= \partial\rho\Big(\sum_{s=1}^M [\chi_s Y_{\varepsilon,P_s}, \overline{L}]\Big) \\
&= -\sum_{s=1}^M \left(\overline{L}\chi_s\right)\partial\rho(Y_{\varepsilon,P_s}) + \sum_{s=1}^M \chi_s \partial\rho\left([Y_{\varepsilon,P_s}, \overline{L}]\right) \\
&= -\sum_{s=1}^M \left(\overline{L}\chi_s\right)\partial\rho(X_\varepsilon) + \sum_{s=1}^M \chi_s O(\varepsilon) = O(\varepsilon).
\end{aligned} \tag{5.72}$$

In the last equality, we have used that $\partial\rho(X_\varepsilon)$ is independent of s (i.e., X_ε is defined 'globally'), so can come out of the sum, and that $\sum_{s=1}^M \overline{L}\chi_s = \overline{L}(\sum_{s=1}^M \chi_s) = \overline{L}(1) = 0$. Also, by construction, $Y_\varepsilon\rho = X_\varepsilon\rho$, so that (i) of Theorem 5.7 is also satisfied. By what was already shown above, these fields can now be extended into a neighborhood of K in $\mathbb{C}^n$ to obtain a family that satisfies the assumptions in Theorem 5.7. □

5.8 Good vector fields and the complex Hessian of a defining function

In view of the previous section, in order to understand when there is a family of vector fields $\{X_\varepsilon\}$ as in Theorem 5.7, it suffices to analyze the conditions in Theorem 5.9. We argue at first locally, near a boundary point P, as in the proof of Theorem 5.9. Set $L_n = (1/|\partial\rho|^2)\sum_{j=1}^{n}(\partial\rho/\partial\bar{z}_j)\partial/\partial z_j$ (so that $L_n\rho = \partial\rho(L_n) = 1$). Then

$$X_\varepsilon = e^{h_\varepsilon}L_n + \sum_{j=1}^{n-1} a_j^\varepsilon L_j, \tag{5.73}$$

where the a_j^ε's and h_ε are smooth. Moreover, h_ε is bounded uniformly in ε and has imaginary part that is $O(\varepsilon)$ (because of (i)). As in (5.68), we find

$$\partial\rho([X_\varepsilon,\overline{L}]) = -e^{h_\varepsilon}\overline{L}h_\varepsilon + e^{h_\varepsilon}\partial\rho([L_n,\overline{L}]) + \sum_{j=1}^{n-1} a_j^\varepsilon \partial\rho([L_j,\overline{L}]), \tag{5.74}$$

where L is a tangential field (say of unit length) of type $(1,0)$. Note that $\partial\rho([L_j,\overline{L}])$, the Levi form (see (2.45)), is semidefinite (Ω is pseudoconvex). Therefore, if we evaluate at a point $Q \in b\Omega$ where $L(Q) \in \mathcal{N}(Q)$, then $\partial\rho([L_j,\overline{L}])(Q) = 0$ for $1 \le j \le n-1$, by the Cauchy–Schwarz inequality. As a result, we obtain, in view of the uniform boundedness of h_ε, the condition

$$-\overline{L}h_\varepsilon(Q) + \partial\rho([L_n,\overline{L}])(Q) = O(\varepsilon), \quad L(Q) \in \mathcal{N}(Q),\ Q \in K. \tag{5.75}$$

Note that h_ε is defined 'globally' (i.e., in a neighborhood of K), because L_n is. Therefore, so is (5.75). Conversely, if there is a uniformly bounded family $\{h_\varepsilon\}$ so that (5.75) holds, then the family $\{X_\varepsilon\} = \{e^{h_\varepsilon}L_n\}$ satisfies the assumptions in Theorem 5.9. Consequently, we have the following fact, which we formulate as a lemma for emphasis.

Lemma 5.10. *The vector fields required in Theorem 5.9 exist if and only if there is a uniformly bounded family of functions $\{h_\varepsilon\}$ satisfying (5.75), with h_ε defined in a neighborhood U_ε of K, and with* $\operatorname{Im} h_\varepsilon = O(\varepsilon)$.

It is trivial to satisfy(5.75) when $\partial\rho([L_n,\overline{L}])(Q) = 0$ for $L(Q) \in \mathcal{N}(Q)$ and $Q \in K$: we may simply take $h_\varepsilon \equiv 0$ for all $\varepsilon > 0$. To see when this might happen, it is useful to have the following expression for $\partial\rho([L_n,\overline{L}])$ (compare the remark after the proof of Lemma 2.5). Set $L = \sum_{k=1}^{n} w_k(\partial/\partial z_k)$. Then

$$\begin{aligned}\partial\rho([L_n,\overline{L}]) &= -\sum_{j=1}^{n}\overline{L}\left(\frac{1}{|\partial\rho|^2}\frac{\partial\rho}{\partial\bar{z}_j}\right)\frac{\partial\rho}{\partial z_j}\\ &= \frac{1}{|\partial\rho|^2}\sum_{j=1}^{n}\frac{\partial\rho}{\partial\bar{z}_j}\overline{L}\left(\frac{\partial\rho}{\partial z_j}\right) = \frac{1}{|\partial\rho|^2}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\frac{\partial\rho}{\partial\bar{z}_j}\overline{w_k}.\end{aligned} \tag{5.76}$$

In the second equality, we have used that $(1/|\partial\rho|^2)\sum_{j=1}^{n}(\partial\rho/\partial\bar{z}_j)(\partial\rho/\partial z_j)$ is constant.

(5.76) shows that $\partial\rho([L_n, \overline{L}])(Q) = 0$ for $L(Q) \in \mathcal{N}(Q)$ does indeed hold for a large class of domains: when the complex Hessian of ρ at points of the boundary is positive semidefinite on all of $\mathbb{C}^n$, that is, when Ω admits a defining function that is plurisubharmonic at the boundary, the Cauchy–Schwarz inequality implies that the right-hand side of (5.76) (hence $\partial\rho([L_n, \overline{L}])$) vanishes for $L(Q) \in \mathcal{N}(Q)$. In fact, it suffices that there is a defining function whose complex Hessian is positive semidefinite on the span of $\mathcal{N}(P)$ and the complex normal at P, at weakly pseudoconvex (or infinite type) boundary points. Thus we obtain the following corollary to Theorem 5.9 ([48]):

Corollary 5.11. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ that admits a defining function whose complex Hessian has the property that at each boundary point P of infinite type, its restriction to the span (over $\mathbb{C}$) of $\mathcal{N}(P)$ and the complex normal at P is positive semidefinite. Then the $\overline{\partial}$-Neumann operators N_q, $1 \le q \le n$, and the Bergman projections P_q, $0 \le q \le n$, are continuous in $W^s_{(0,q)}(\Omega)$ when $s \ge 0$.*

The main application of Corollary 5.11 is to domains that admit a defining function which is plurisubharmonic at the boundary. Such a domain is in particular pseudoconvex, but pseudoconvexity is slightly weaker: it requires only that the Hessian be positive semidefinite on the complex tangent space, rather than in all directions. The reader should note that whether or not the complex Hessian is positive semidefinite at points of the boundary depends on the defining function, in contrast to definiteness on the complex tangent space.

Every smooth convex domain admits a smooth defining function that is plurisubharmonic at the boundary, namely its Minkowski functional ([261], p. 35). The Minkowski functional is convex in a neighborhood of the boundary, hence plurisubharmonic there (it is convex, hence subharmonic, on each complex line), and it is smooth when Ω is. Nonconvex examples may be obtained by considering sublevel sets of sums of moduli of holomorphic functions. In addition, the superlevel sets $\Omega_\delta = \{z \in \Omega \mid \operatorname{dist}(z, b\Omega) > \delta\}$ of the boundary distance in a smooth bounded pseudoconvex domain admit a defining function that is plurisubharmonic at the boundary. Indeed, $\rho_\delta(z) := -\log(\operatorname{dist}(z, b\Omega)) + \log\delta$ is such a defining function: pseudoconvexity of Ω implies that $-\log(\operatorname{dist}(z, b\Omega))$ is plurisubharmonic in Ω (cf. [207], [259]), and for δ small enough, its gradient does not vanish on $b\Omega_\delta$. In particular, then, when the *outside* level sets of the boundary distance are pseudoconvex, Ω admits a defining function that is plurisubharmonic. Such domains are studied in [302] under the name ‘special domains of holomorphy’.

The class of domains that admit a defining function that is plurisubharmonic at the boundary is also invariant under biholomorphisms: if Ω_1 and Ω_2 are two smooth bounded pseudoconvex domains that are biholomorphic, and if Ω_1 admits a defining function that is plurisubharmonic at the boundary, then so does Ω_2. The reason is the same as in Remark 1 following Corollary 5.8: Corollary 5.11 and Bell’s regularity theorem ([23]) combine to show that the biholomorphism extends to a diffeomorphism of the closures; it then suffices to push forward the defining function of Ω_1 (and extend it smoothly to a neighborhood of $\overline{\Omega_2}$).

Examples of pseudoconvex domains that do not admit a defining function that is plurisubharmonic at the boundary are the so called worm domains, which we will describe later (see Lemma 5.20 below). There are also examples that do not admit even local defining functions that are plurisubharmonic at the boundary ([126], [22]).

One can obtain domains that satisfy the assumptions in Corollary 5.11, but that do not admit a defining function that is plurisubharmonic at the boundary by considering domains having as suitable lower dimensional sections domains without local defining functions that are plurisubharmonic at the boundary (from the previous paragraph). In the context of vector fields, this observation comes from Remark 3 in [51].

Remarks. (i) A Sobolev-$1/2$ estimate for the Bergman projection P_0 on C^3-domains that admit a plurisubharmonic defining function had been shown earlier in [56]. Sobolev estimates for the $\bar{\partial}$-Neumann problem on convex domains in $\mathbb{C}^2$ were established in [80]. A $1/2$ estimate on domains with a plurisubharmonic Lipschitz defining function is proven in [226].

(ii) It is shown in [169] that if Ω admits a defining function whose Hessian has the property that at each boundary point, the sum of any q eigenvalues is nonnegative, then the Bergman projections $P_{q-1}, \ldots, P_n$ are (exactly) globally regular. When $q = 1$, this paper gives another approach to Corollary 5.11. The reader should notice the analogy to the scale of the successively weaker properties (P_q) from Chapter 3. We will elaborate further on this later in the chapter; see in particular Lemma 5.23 and Corollary 5.25 below.

(iii) Although not every domain admits a plurisubharmonic defining function, there always exists a defining function ρ and η, $0 < \eta \leq 1$, such that $-(-\rho)^\eta$ is plurisubharmonic in Ω (near the boundary), see [106], [256]. This can be exploited to obtain estimates in the Sobolev scale up to a certain level that depends on η. Such a quantitative analysis is carried out in [196], from the point of view of microlocal analysis. For small Sobolev levels, a somewhat more precise range can be obtained in terms of the Diederich–Fornæss exponent $\eta(\Omega)$ of the domain (the supremum of the exponents η from the first sentence). This exponent has recently been shown in [158] to be strictly positive when the domain is only assumed to have Lipschitz boundary. The author then observes firstly that his result combines with [37] to give estimates in W^s for the Bergman projections P_q and the canonical solution operators $\bar{\partial}^* N_q$, $1 \leq q \leq n$, for $0 \leq s < \eta(\Omega)/2$, and secondly that the work in [60] can be adapted to also give estimates for the $\bar{\partial}$-Neumann operators N_q for the same Sobolev range. This result is sharp with respect to the assumption of Lipschitz regularity of the boundary in the following sense. There exists a bounded pseudoconvex domain Ω in $\mathbb{C}^2$, smooth except at one point, such that the $\bar{\partial}$-Neumann operator is continuous on $W^s_{(0,1)}(\Omega)$ for no $s > 0$ ([264]).

(iv) Suppose ρ is a defining function for Ω that is plurisubharmonic at the boundary. It is interesting to see what the implications are for the behavior of the Hessian of ρ *near* the boundary of Ω. This question has recently been studied in [127], [128]. ρ need not be plurisubharmonic in any neighborhood of the boundary. But the authors obtain estimates on the Hessian of ρ which imply that for every $\eta > 1$, one can produce a new defining function r such that r^η is (strictly) plurisubharmonic in $\mathbb{C}^n \setminus \overline{\Omega}$ (near $b\Omega$). This implies in particular that $\overline{\Omega}$ admits a neighborhood basis consisting of pseudoconvex sets (a Stein neighborhood basis): the sublevel sets of r^η will do. Whether or not Ω admits a defining function that is plurisubharmonic in a neighborhood of the boundary remains open.

(v) As we pointed out, Corollary 5.11 applies to convex domains. However, in this case it is not necessary to proceed via a plurisubharmonic defining function (the domain's Minkowski functional). It was observed in [237] that derivatives of the unit normal in Levi null directions vanish on the boundary of a convex domain. This shows directly that one can take the complex unit normal as X_ε in Theorem 5.9. A quick way to see that these derivatives vanish is via expressing the Levi form in terms of the second fundamental form of the boundary: $L_\rho(X,\overline{X}) = \Pi(X,X) + \Pi(JX,JX)$, with the usual identifications, and with $JX = iX$ (componentwise). When $L_\rho(X,\overline{X}) = 0$, both $\Pi(X,X)$ and $\Pi(JX,JX)$ must vanish, since convexity means that the second fundamental form is positive semidefinite. This translates into the statement about the derivatives of the unit normal. Details may be found in [54], Theorem 2 in Section 10.3, and the discussion following it.

Here is another situation where there exists a defining function ρ that satisfies $\partial\rho([L_n,\overline{w}])(Q) = 0$ when $w \in \mathcal{N}(Q)$; it comes from [284].

Corollary 5.12. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with the property that for each pair $(z,w) \in b\Omega \times \mathbb{C}^n$ with $w \in \mathcal{N}(z)$, there exists a sequence $\{(z_j,w_j\}_{j=1}^\infty$ in $\Omega \times \mathbb{C}^n$ such that $(z_j,w_j) \to (z,w)$ as $j \to \infty$, and for all j, w_j is in the null space of the Levi form at z_j of the level set of the boundary distance through z_j. Then the $\overline{\partial}$-Neumann operators N_q, $1 \le q \le n$, and the Bergman projections P_q, $0 \le q \le n$, are continuous on $W^s_{(0,q)}(\Omega)$ when $s \ge 0$.*

The condition in Corollary 5.12 says that weakly pseudoconvex directions at boundary points should be limits of weakly pseudoconvex directions to level sets of the boundary distance. It arose in [284] in connection with domains whose closure admits a 'very nice' Stein neighborhood basis. As far as the author can see, it is not currently well understood. It does, however, also cover convex domains (see Remark 4 in [284]).

Proof of Corollary 5.12. We only have to show that $\partial\rho([L_n,\overline{L}])(Q) = 0$ when $L(Q) \in \mathcal{N}(Q)$. Choose the signed boundary distance for ρ and use again that $-\log(-\rho)$ is plurisubharmonic in Ω, i.e., that its Hessian is positive semidefinite. This gives, after multiplication by the (positive) factor $-\rho(z)$,

$$\sum_{j,k=1}^{n}\left(\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(z) - \frac{1}{\rho(z)}\frac{\partial\rho}{\partial z_j}(z)\frac{\partial\rho}{\partial\bar{z}_k}(z)\right)w_j\overline{w_k} \ge 0, \quad z \in \Omega,\ w \in \mathbb{C}^n. \tag{5.77}$$

Now let (z,w) as in the corollary, $(z_l,w_l) \to (z,w)$, and let $L_n(z) = (\zeta_1(z),\dots,\zeta_n(z))$ (note that L_n is defined in a neighborhood of the boundary). Denote by $\mathcal{Q}(w,\overline{w})$ the sesquilinear form on the left-hand side of (5.77). (5.77) says that $\mathcal{Q}$ is positive semidefinite. This implies, for l sufficiently big,

$$\left|\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k}(z_l)\zeta_j(z_l)\overline{(w_l)_k}\right| = |\mathcal{Q}(z_l)(\zeta(z_l), \overline{w_l})|$$
$$\leq \mathcal{Q}(z_l)(\zeta(z_l), \overline{\zeta(z_l)})^{1/2}\mathcal{Q}(z_l)(w_l, \overline{w_l})^{1/2} = 0, \tag{5.78}$$

where the inequality in the middle is the Cauchy–Schwarz inequality for $\mathcal{Q}$. The last equality holds because $\mathcal{Q}(z_l)(w_l, \overline{w_l}) = 0$ (w_l is complex tangential to the level surface of ρ through z_l and is a Levi null direction). Letting l tend to infinity completes the proof of Corollary 5.12. ☐

Remark. We have stated Corollary 5.12 as it is for simplicity. The above proof gives a little more. Namely, $Q(z_l)(\zeta(z_l), \overline{\zeta(z_l)}) \approx 1/\rho(\zeta_l)$. Therefore, w_l does not necessarily have to be a Levi null direction to the level surface; it suffices that the complex Hessian $\sum_{j,k=1}^{n}(\partial^2\rho/\partial z_j\partial\bar{z}_k)(z_l)(w_l)_j\overline{(w_l)_k}$ (which equals $Q(z_l)(w_l, \overline{w_l})$) tends to zero faster than then the boundary distance of z_l.

5.9 A useful one-form

In order to analyze the condition (5.75) when $\partial\rho([L_n, \bar{L}])(Q)$ is not necessarily zero when $L(Q) \in \mathcal{N}(Q)$, we observe that this quantity depends only on $L(Q)$, and linearly so (terms that contain derivatives of the components of L are of type $(0, 1)$, thus vanish once $\partial\rho$ is applied). That is, this quantity represents a 1-form. We study this form following D'Angelo ([87], [91]). Everything that is needed from the calculus of differential forms may be found in [227], Chapter 2, or [152], Chapter IV. Following the notation of [91], we let η denote a purely imaginary, nonvanishing one-form on $b\Omega$ that annihilates $T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$. For example, $\eta = (1/2)(\partial\rho - \bar{\partial}\rho)$, where ρ is a defining function for Ω, will work. Let T be the (unique) purely imaginary tangential vector field orthogonal to $T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$ and such that $\eta(T) \equiv 1$ on $b\Omega$. With the previous choice of η, T becomes $T = L_n - \overline{L_n}$ (L_n as above).

The form we are interested in is

$$\alpha = -\mathcal{L}_T\eta, \tag{5.79}$$

where $\mathcal{L}_T$ denotes the Lie derivative in the direction of T. We suppress the dependence on η in the notation; it will turn out that the relevant properties of α will all be independent of the choice of η. Note that because both η and T are purely imaginary, α is real.

To see the connection with the form that occurs in (5.75), let us compute $\alpha(\bar{L})$, where L is a (local) section of $T^{(1,0)}(b\Omega)$. The definition of the Lie derivative and the fact that η annihilates $T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$ imply that

$$\alpha(\bar{L}) = -\mathcal{L}_T\eta(\bar{L}) = -T\big(\eta(\bar{L})\big) + \eta\big([T, \bar{L}]\big) = \eta\big([T, \bar{L}]\big) = \eta(aT) = a, \tag{5.80}$$

where $[T, \overline{L}] = aT \mod T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$, i.e., $a = \alpha(\overline{L})$ is the T-component, mod $T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$, of $[T, \overline{L}]$. The same computation applies to $\alpha(L)$ (alternatively, since α is real, one can just take conjugates). With the particular choice of η and T from above, we get from (5.80) that

$$\alpha(\overline{L}) = \frac{1}{2}\big(\partial\rho - \bar{\partial}\rho\big)\big([L_n - \overline{L_n}, \overline{L}]\big) = \partial\rho\big([L_n - \overline{L_n}, \overline{L}]\big) = \partial\rho\big([L_n, \overline{L}]\big). \tag{5.81}$$

Here we have used first that $\big(\partial\rho + \bar{\partial}\rho\big)\big([L_n - \overline{L_n}, \overline{L}]\big) = d\rho\big([L_n - \overline{L_n}, \overline{L}]\big) = 0$ ($[L_n - \overline{L_n}, \overline{L}]$ is tangential), and then that $\partial\rho\big([\overline{L_n}, \overline{L}]\big) = 0$ (since $\partial\rho$ annihilates vectors of type $(0,1)$). So $\alpha(\overline{L})$ is precisely the 1-form from above, i.e., the quantity that occurs in (5.75). The first part of the following Proposition is therefore just a restatement of Lemma 5.10.

Proposition 5.13. *A family of vector fields as in Theorem 5.9 (and hence one as in Theorem 5.7) exists if and only if there exists a family $\{h_\varepsilon\}$ of smooth functions defined in a neighborhood (in $b\Omega$) U_ε of K, uniformly bounded in ε, with* $\operatorname{Im} h_\varepsilon = O(\varepsilon)$, *satisfying*

$$dh_\varepsilon(\overline{L})(Q) = \alpha(\overline{L})(Q) + O(\varepsilon), \quad L(Q) \in \mathcal{N}(Q),\ Q \in K. \tag{5.82}$$

Moreover, whether or not such a family $\{h_\varepsilon\}$ exists is intrinsic: it does not depend on the choice of the form η that determines α; in particular, it does not depend on the choice of defining function in (5.81).

Here dh_ε denotes the differential of h_ε. Note that $dh_\varepsilon(\overline{L}) = \bar{\partial}h_\varepsilon(\overline{L})$ and $\alpha(\overline{L}) = \alpha_{(0,1)}(\overline{L})$ ($\alpha_{(0,1)}$ is the $(0,1)$-part of α), so that (5.82) can be reformulated as

$$\bar{\partial}h_\varepsilon(\overline{L})(Q) = \alpha_{(0,1)}(\overline{L})(Q) + O(\varepsilon), \quad L(Q) \in \mathcal{N}(Q),\ Q \in K. \tag{5.83}$$

Proof of Proposition 5.13. We only have to prove the second part of the proposition. Since the defining function does not matter in Theorem 5.9, existence of the family $\{h_\varepsilon\}$ cannot depend on the choice of defining function in (5.81). To understand the intrinsic nature of (5.82), and to see that a general form η can be allowed, let $\tilde{\eta} = e^g\eta$, where g is a smooth real valued function. (The case $\tilde{\eta} = -e^g\eta$ is analogous; these are the only two cases, as both η and $\tilde{\eta}$ are purely imaginary and nonvanishing.) Then $\widetilde{T} = e^{-g}T$, and for $X \in T^{(1,0)}(b\Omega)$, we obtain in view of equation (5.80) (and the remark following its statement)

$$\begin{aligned}\tilde{\alpha}(X) &= \tilde{\eta}([\widetilde{T}, X]) = e^g\eta([e^{-g}T, X]) \\ &= \eta([T, X]) + Xg = \alpha(X) + dg(X).\end{aligned} \tag{5.84}$$

In the third equality, we have used that $\eta(T) \equiv 1$. Because α, $\tilde{\alpha}$, and g are real, (5.84) also holds with X replaced by $\overline{X}$. So $\tilde{\alpha}$ and α differ by dg on $T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$, and if $\{h_\varepsilon\}$ satisfies (5.82) with α, then $\{\widetilde{h_\varepsilon}\} := \{h_\varepsilon + g\}$ satisfies (5.82) with $\tilde{\alpha}$. ☐

Note that combining (5.81) and (5.76) gives the expression for $\alpha(\overline{L})$ directly in terms of the defining function ρ. It is, up to normalization, the Hessian of the defining function applied to the normal and $\overline{L}$ (the 'mixed' term in the Hessian):

$$\alpha(\overline{L}) = \frac{1}{|\partial\rho|^2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \frac{\partial \rho}{\partial \bar{z}_j} \overline{w_k}. \tag{5.85}$$

Remark. The equivalence of the conditions in Theorems 5.7 and 5.9 and Proposition 5.13 were studied in [288], where they were shown to be equivalent to the existence of a family of 'essentially pluriharmonic defining functions' and to the existence of a family of conjugate normals which are 'approximately holomorphic in weakly pseudoconvex directions'. We refer the reader to the main theorem in [288] for details. We also take this opportunity to correct an inaccuracy in [288]. Namely, the authors used that the differential dh_ε in equation (12) there is real on K. This does not follow in general when h_ε is known to be real only on K. To correct the situation, one should assume that condition (1) in [288], $C^{-1} < X_\varepsilon\rho < C$, holds not just on K, but in a neighborhood U_ε of K (the point being that $X_\varepsilon\rho$ is *real* in a full neighborhood). It is not clear how much more stringent this condition actually is in practice; we discuss this in Remark (iii) following the statement of Theorem 5.22 below. Alternatively, one can modify the definitions of 'α is approximately exact on the null space of the Levi form' and of a family of 'essentially pluriharmonic defining functions' in a fairly obvious way, in analogy to Proposition 5.13 and Theorem 5.22, respectively.

In Corollaries 5.11 and 5.12, the domain has a defining function so that the resulting form α vanishes on the null space of the Levi form. This allows for the trivial solution $h_\varepsilon \equiv 0$ ($\varepsilon > 0$) in (5.82) in Proposition 5.13. When α is not (known to be) zero, the following closedness property is important (in light of (5.82)): $d\alpha$ vanishes on $\mathcal{N}_P \oplus \overline{\mathcal{N}_P}$. This property was discovered in [51].

Lemma 5.14. *Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^n$, $P \in b\Omega$. Denote by $\mathcal{N}_P \subset T_P^{(1,0)}(b\Omega)$ the null space of the Levi form at P. If X and Y belong to $\mathcal{N}_P \oplus \overline{\mathcal{N}_P}$, then $(d\alpha)_P(X, Y) = 0$.*

Proof. We follow [51]. Extend X and Y to local sections of $T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$. Using that the Lie derivative and the exterior derivative commute, and the definition of the Lie derivative on 2-forms, we obtain

$$-(d\alpha)(X, Y) = \mathcal{L}_T d\eta(X, Y) = T d\eta(X, Y) - d\eta([T, X], Y) - d\eta(X, [T, Y]). \tag{5.86}$$

By definition of the exterior derivative, we have

$$d\eta([T, X], Y) = [T, X]\eta(Y) - Y\eta([T, X]) - \eta([[T, X], Y]). \tag{5.87}$$

The first term on the right-hand side of (5.87) vanishes because $\eta(Y) \equiv 0$. In the third term, write $[T, X] = \eta([T, X])T + A + \overline{B}$, where $A, B \in T^{(1,0)}(b\Omega)$. Then

$[A, Y] \in T^{(1,0)}(b\Omega)$, whence $\eta([A, Y]) \equiv 0$. Write $Y = Y_1 + \overline{Y_2}$, with $Y_1, Y_2 \in T^{1,0}(b\Omega) \oplus T_{0,1}(b\Omega)$. Then $\eta([\overline{B}, \overline{Y_2}]) \equiv 0$, and

$$\begin{aligned}\eta\left([\overline{B}, Y]\right) &= \eta\left([\overline{B}, Y_1]\right) = -\eta\left([Y_1, \overline{B}]\right) \\ &= -g\left(\partial\rho - \bar{\partial}\rho\right)\left([Y_1, \overline{B}]\right) = -2g\partial\rho\left([Y_1, \overline{B}]\right),\end{aligned} \tag{5.88}$$

where g is a nonvanishing function. We have used that $(\partial\rho + \bar{\partial}\rho)([Y, \overline{B}]) = 0$ (both Y and $\overline{B}$ are tangential, hence so is their commutator). The right-hand side of (5.88) is a multiple of the Levi form (see (2.45)). Because Ω is pseudoconvex (so that the Levi form is positive semidefinite, and Cauchy–Schwarz applies) and $Y_1(P) \in \mathcal{N}_P$, this term must vanish at P. Consequently, the third term in (5.87) equals, at P, $\eta(\eta([T, X])T, Y])(P) = \eta([T, X])(P)\eta([T, Y])(P) - Y\eta([T, X])(P)\eta(T)(P)$. Inserting this last expression into (5.87) and noting that $\eta(T)(P) = 1$ gives

$$d\eta([T, X], Y)(P) = -\eta([T, X])(P)\eta([T, Y])(P). \tag{5.89}$$

A similar computation, using that $X(P) \in T_P^{(1,0)}(b\Omega) \oplus T_P^{(0,1)}(b\Omega)$, gives

$$d\eta(X, [T, Y])(P) = \eta([T, X])(P)\eta([T, Y])(P). \tag{5.90}$$

Thus, upon inserting (5.89) and (5.90) into (5.86):

$$d\alpha(X, Y)(P) = -Td\eta(X, Y)(P) = T\eta([X, Y])(P); \tag{5.91}$$

the last equality in (5.91) follows from the definition of the exterior derivative and the fact that $\eta(X) \equiv \eta(Y) \equiv 0$.

We use (5.91) to see that $d\alpha(X, Y)(P) = 0$. First, let both X and Y be in $T^{(1,0)}(b\Omega)$. Then $[X, Y] \in T^{(1,0)}(b\Omega)$; so, $\eta([X, Y]) \equiv 0$, and $T\eta([X, Y])(P) = 0$. Since α is real, this also holds when both X and Y belong to $T^{(0,1)}(b\Omega)$. Next, consider the case where $X \in T^{(1,0)}(b\Omega)$ and $Y = \overline{Z} \in \overline{T^{(1,0)}(b\Omega)}$. The form $d\alpha([X, \overline{Z}])(P)$ is skew hermitian on $\mathcal{N}_P$. In order to see that it vanishes, it suffices, by polarization, to see that the associated sesquilinear form $d\alpha([Z, \overline{Z}])(P) = T\eta([Z, \overline{Z}])(P)$ vanishes. Because Ω is pseudoconvex and $Z(P) \in \mathcal{N}_P$, the function $\eta([Z, \overline{Z}])$ (the Levi form, up to a nonvanishing factor, see (5.88)) takes a local extremum at P, whence $T\eta([Z, \overline{Z}])(P) = 0$. It now follows from linearity that $d\alpha(X, Y) = 0$ for all X and Y in $\mathcal{N}_P \oplus \overline{\mathcal{N}_P}$. The proof of the lemma is complete. □

5.10 Submanifolds in the boundary and a cohomology class

Lemma 5.14 implies that when M is a submanifold of the boundary with the property that $T_P(M) \subseteq \mathcal{N}_P$ for all P in M (for example when M is a complex submanifold of the boundary), then $\alpha|_M$, the restriction of α to M, is closed and so defines a de Rham cohomology class $[\alpha|_M]$ on M. This class is intrinsic:

Lemma 5.15. *Let M be a submanifold of $b\Omega$ such that at each point P of M, the (real) tangent space to M is contained in $\mathcal{N}_P$. The class $[\alpha|_M]$ does not depend on the choice of η.*

Proof. We have shown in the proof of Proposition 5.13 that two forms associated to two different choices of η differ on $T^{(1,0)}(b\Omega) \oplus T^{(0,1)}(b\Omega)$, hence on M, by the differential of a smooth function (see (5.84)). Consequently, they define the same cohomology class on M. □

Remark. When M is a complex submanifold of the boundary, the closedness of $\alpha|_M$ corresponds to the pluriharmonicity of certain argument functions, as in [14], [17], Proposition 3.1, [18], Lemma 1, and [105], page 290. The class $[\alpha|_M]$ occurs in [17] in connection with sufficient conditions for the existence of a Stein neighborhood basis of the closure of Ω: if $[\alpha|_M]$ is below a certain threshold in a suitable norm on cohomology, then a Stein neighborhood basis exists. The authors interpret the class geometrically as a measure of how much the boundary winds around M. This interpretation, in terms of winding of the normal, is also implicit in [105].

The main difficulty in a general study of (5.82) stems from the fact that the dimension of $\mathcal{N}(P)$ in general varies with $P \in b\Omega$. However, this fact is not a problem when the boundary points of infinite type are contained in a submanifold M of the boundary as in Lemma 5.15, and the following theorem from [51] is another corollary to Theorems 5.7 and 5.9. It furnishes easily verifiable geometric conditions that imply global regularity (especially the version given in Corollary 5.17). We denote by $H^1(M)$ the first de Rham cohomology of M.

Corollary 5.16. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose there is a smooth real submanifold M (with or without boundary) of $b\Omega$ that contains all the points of infinite type of $b\Omega$ and whose real tangent space at each point is contained in the null space of the Levi form at that point. If the $H^1(M)$ cohomology class $[\alpha|_M]$ is zero (i.e., α is exact on M), then the $\bar{\partial}$-Neumann operators N_q, $1 \le q \le n$, and the Bergman projections P_q, $0 \le q \le n$, are continuous in the Sobolev space $W^s_{(0,q)}(\Omega)$ when $s \ge 0$.*

When M is a manifold with boundary, we assume that it is smooth as a manifold with boundary. We need not distinguish between the cohomology of M and that of its interior (in our situation): these are naturally isomorphic (see for example [152], p. 231). Note that when M is a complex submanifold of the boundary, then it has to be closed (since all points in the closure are points of infinite type). Therefore, M has to be a manifold with boundary (otherwise, all holomorphic functions, in particular the coordinate functions in $\mathbb{C}^n$, would be constant). The class $[\alpha|_M]$ is always trivial when $H^1(M)$ is trivial, for example when the connected components of M are simply connected. The appearance of the class $[\alpha|_M]$ explains why an annulus in the boundary is bad for Sobolev estimates on the worm domains (see Theorem 5.21), but not on some other Hartogs domains ([49]), while an analytic disc is always benign ([49]).

For emphasis, we formulate the following special case of Corollary 5.16 separately.

Corollary 5.17. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, of finite type except for a smooth (as a manifold with boundary) simply connected complex manifold in the boundary. Then the $\bar{\partial}$-Neumann operators N_q, $1 \le q \le n$, and the*

Bergman projections P_q, $0 \le q \le n$, are continuous in the Sobolev space $W^s_{(0,q)}(\Omega)$ when $s \ge 0$.

Proof of Corollary 5.16. Choose a defining function ρ; this gives α. Since $[\alpha|_M] = 0$, there is a real-valued function $h \in C^\infty(M)$ (smooth up to the boundary of M if M is a manifold with boundary) with

$$d_M h = \alpha|_M. \tag{5.92}$$

Fix a point $P \in K$ (P may be in the boundary of M). Choose local sections $L_1, \dots, L_{n-1}$ of $T^{(1,0)}(b\Omega)$ such that $L_1, \dots, L_m$ span the complex tangent space to M at P, and $L_1(P), \dots, L_{n-1}(P)$ is a basis for the complex tangent space to $b\Omega$ at P. Here, $0 \le m \le (n-1)$; if M has no complex tangents at P, then $m = 0$. ($L_1, \dots, L_m$ need not be tangent to M at points other than P.) When $m + 1 \le j \le n - 1$, the real two dimensional plane spanned by the real and imaginary parts of $L_j(P)$ intersects the real tangent space to M at P in a point or in a line (since it is not contained in this real tangent space). Therefore, we can extend h from M (near P) into a neighborhood (in $\mathbb{C}^n$) of P to a function $\widetilde{h}_P$ that satisfies

$$\overline{L_j}\widetilde{h}_P(P) - \alpha(\overline{L_j})(P) = 0, \quad 1 \le j \le n. \tag{5.93}$$

(L_n is the complex normal, as before.) Indeed, it suffices to extend h with appropriately prescribed derivatives at P in the directions transverse to M. This extension can be chosen so that its modulus does not exceed $2\max_{z \in K}|h(z)|$. Given $\varepsilon > 0$ fixed, $\widetilde{h}_P$ satisfies

$$\left|\overline{L_j}\widetilde{h}_P(Q) - \alpha(\overline{L_j})(Q)\right| < \varepsilon, \quad Q \in U_P,\ 1 \le j \le n, \tag{5.94}$$

where U_P is a small enough neighborhood (in $\mathbb{C}^n$) of P. We can over the compact set K of points of infinite type by finitely many of these neighborhoods, say $U_{P_1}, \dots, U_{P_S}$. Denote by $\{\varphi_j\}_{j=1}^S$ a partition of unity of a neighborhood of K subordinate to the cover $U_{P_1}, \dots, U_{P_S}$. Set

$$h_\varepsilon = \sum_{j=1}^{S} \varphi_j \widetilde{h}_{P_j}. \tag{5.95}$$

h_ε is defined in a (full) neighborhood of K, and $|h_\varepsilon| \le 2\max_{z \in K}|h(z)|$. Let $Q \in K$, and L a $(1,0)$-field on $b\Omega$, near Q. Then

$$\overline{L}h_\varepsilon(Q) = \sum_{j=1}^{S}\left(\varphi_j(Q)\overline{L}\widetilde{h}_{P_j}(Q) + \overline{L}\varphi_j(Q)\widetilde{h}_{P_j}(Q)\right). \tag{5.96}$$

But $Q \in K \subseteq M$, so that $\widetilde{h}_{P_j}(Q) = h(Q)$ is independent of j. Therefore,

$$\sum_{j=1}^{S} \overline{L}\varphi_j(Q)\widetilde{h}_{P_j}(Q) = h(Q)\sum_{j=1}^{S}\overline{L}\varphi_j(Q) = 0 \tag{5.97}$$

(because $\sum_{j=1}^{S}\varphi_j = 1$ in a neighborhood of K). Therefore,

$$\left|\overline{L}h_\varepsilon(Q) - \alpha(\overline{L}(Q)\right| \leq \sum_{j=1}^{S}\varphi_j(Q)\left|\overline{L}\widetilde{h}_{P_j}(Q) - \alpha(\overline{L})(Q)\right| < \varepsilon, \tag{5.98}$$

by (5.94). If we choose the neighborhood U_ε of K small enough, then the family h_ε that we have constructed satisfies (5.82) in Proposition 5.13 (in fact, it is not even required that $L(Q) \in \mathcal{N}(Q)$), and the proof of Corollary 5.16 is complete. □

Remarks. (i) In the situation of Corollary 5.17, the cohomology class $[\alpha|_M]$ actually is an obstruction to the existence of a family $\{h_\varepsilon\}$ as in Proposition 5.13. If such a family exists, then by the uniform bounds, a suitable subsequence of $\{h_\varepsilon\}$ converges weakly in, say, $\mathcal{D}'(M)$ to a real distribution h as $\varepsilon \to 0^+$, with $\bar{\partial}_M h = \alpha_{(0,1)}|_M$ (the $(0,1)$-part of α). Because both α and h are real, $d_M h = (\partial_M + \bar{\partial}_M)h = \alpha_{(1,0)}|_M + \alpha_{(0,1)}|_M = \alpha|_M$, and $[\alpha|_M] = 0$ (h is automatically in $C^\infty(M)$ because α is). Whether or not $[\alpha|_M] \neq 0$ also excludes global regularity is open. There is a result in [12] that points in this direction. Roughly speaking, on a class of domains in $\mathbb{C}^2$ with an annulus M in the boundary, if $[\alpha|_M]$ lies outside a certain exceptional set that is small in a potential theoretic sense, then N_1 does not preserve the Sobolev spaces. ($[\alpha|_M] \neq 0$ being an obstruction to regularity would correspond to the exceptional set consisting of the zero class.)

(ii) How profitably a limiting argument as in the previous remark can be used in the general case of Theorem 5.9 to replace the family $\{X_\varepsilon\}$ by a single 'vector field' X on K remains to be seen. Compare also the discussion following the statement of Theorem 5.7.

(iii) The functions $\widetilde{h}_P$, and consequently the functions h_ε, in the proof of Corollary 5.16 can be taken real valued: because α is real and h is real on M, the derivatives that need to be prescribed in directions transverse to M (in view of (5.93)) take real values.

5.11 A foliation in the boundary

In Corollary 5.16, forcing all the action onto the submanifold M took care of the problem that in general, the dimension of $\mathcal{N}$ will vary with the point. One can similarly 'tame' (5.82) by assuming that the set of infinite type points is foliated by complex manifolds. This leads to interesting connections with questions that are studied in foliation theory. To illustrate these connections, we discuss the case of a codimension one foliation in the boundary. Let us assume that Ω is a smooth bounded pseudoconvex domain which is strictly pseudoconvex except at points of K, and that the relative interior of K is foliated by complex hypersurfaces. This foliation is often referred to as the Levi foliation of K. In analogy to the situation in Corollaries 5.16 and 5.17, we look for a real valued smooth function h on K that satisfies

$$dh|_{\mathcal{L}} = \alpha|_{\mathcal{L}} \tag{5.99}$$

for all leaves $\mathcal{L}$ of the Levi foliation. This problem is equivalent to a question well studied in foliation theory, namely whether or not a given codimension one foliation can be defined *globally* by a *closed* 1-form. A foliation is said to be defined by a form if the tangent space to the leaf coincides with the null space of the form. Background on foliation theory and notions used here can be found in the monographs [59], [296], [297]; a concise review in our context is in [7]. We set $\tilde{\eta} = i\eta$ and $\tilde{T} = -iT$ (η and T as above). Then $\tilde{\eta}$ and $\tilde{T}$ are real.

Lemma 5.18. (5.99) *is solvable* (*say on the relative interior of* K) *if and only if the Levi foliation of* K *can be defined by a closed* 1*-form. In this case, if* h *solves* (5.99), $\omega = e^{-h}\tilde{\eta}$ *is such a one form, and vice versa.*

Proof. The Levi foliation on K can be defined by $\tilde{\eta}$: the tangent spaces to the leaves are given by the null space of $\tilde{\eta}$. The Frobenius condition then reads $d\tilde{\eta} \wedge \tilde{\eta} = 0$ (see for example [296], Proposition 2.1). Therefore, $d\tilde{\eta}$ is a 'multiple' of $\tilde{\eta}$: there is a 1-form β such that $d\tilde{\eta} = \beta \wedge \tilde{\eta}$. It turns out that $\alpha = -\mathcal{L}_T\eta = -\mathcal{L}_{\tilde{T}}\tilde{\eta}$ is such a form, that is

$$d\tilde{\eta} = \alpha \wedge \tilde{\eta} \quad \text{on } K, \tag{5.100}$$

(see for example [296], Propositions 2.1 and 2.3). If ω is another 1-form defining the foliation, then $\omega = e^{-h}\tilde{\eta}$ (up to sign), and

$$d\omega = e^{-h}(-dh \wedge \tilde{\eta} + d\tilde{\eta}) = e^{-h}(-dh + \alpha) \wedge \tilde{\eta}, \tag{5.101}$$

in view of (5.100). Therefore,

$$d\omega = 0 \iff (-dh + \alpha) \wedge \tilde{\eta} = 0 \iff (-dh + \alpha)|_{\mathcal{L}} = 0 \tag{5.102}$$

for each leaf $\mathcal{L}$. □

Remark. To assume that a boundary patch is foliated by complex manifolds is equivalent to assuming that the dimension of $\mathcal{N}(Q)$ does not vary (equivalently: the rank of the Levi form is constant): the Jacobi identity $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$ gives that in this case the distribution $\mathcal{N}$ (more precisely, the underlying real distribution in $\mathbb{C}^n \approx \mathbb{R}^{2n}$) is integrable (see [133] for details), and the Frobenius theorem then gives a foliation by leaves whose tangent space at a point Q coincides with $\mathcal{N}(Q)$. This means that all tangents to a leaf are complex tangents; such submanifolds of $\mathbb{C}^n$ are necessarily complex analytic submanifolds ([6], Proposition 1.3.14). For more information on Levi foliations, see [7], Chapter 3.

Lemma 5.18 allows us to bring tools from foliation theory to bear on the problem of solving (5.99), and thus to obtain still further corollaries of Theorems 5.7 and 5.9. One has to have some control over the boundary behavior of the leaves of the Levi foliation. A simple geometric setup that provides this control results when one assumes that the Levi foliation of K is part of a bigger foliation of a Levi flat hypersurface whose intersection with $b\Omega$ is K. The following theorem comes from [132]. A codimension one foliation is called simple if through every point there exists a local transversal line that meets each leaf at most once.

Corollary 5.19. *Let $\Omega \subseteq \mathbb{C}^n$ be a smooth bounded pseudoconvex domain such that the set K of all boundary points of infinite type is the closure of its relative interior in $b\Omega$. Assume K is contained in a smooth Levi flat* (*open*) *hypersurface M, whose Levi foliation satisfies one* (*hence both*) *of the following equivalent conditions:* (i) *the leaves of the restriction of the foliation to a neighborhood of K are topologically closed;* (ii) *the foliation is simple in neighborhood of K. Then the $\bar{\partial}$-Neumann operators N_q, $1 \leq q \leq n$, and the Bergman projections P_q, $0 \leq q \leq n$, are continuous in $W^s_{(0,q)}(\Omega)$, $s \geq 0$.*

Proof. That (i) and (ii) are equivalent is shown in [132], at the beginning of the proof of their Proposition 1.2 in Section 4. The point of this proposition is then that these (equivalent) conditions (among some others) are sufficient for the Levi foliation of M, restricted to an open neighborhood V of the compact set K, to be defined by a closed 1-form. Lemma 5.18 now gives a smooth real valued function h on V so that (5.99) holds. We can extend h from V into an open subset of $\mathbb{C}^n$ which is a neighborhood of K; call the resulting function $\tilde{h}$. $\tilde{h}$ satisfies (5.82) in Proposition 5.13: if $Q \in K$ and $L \in T^{1,0}(b\Omega)$ near Q, then $L(Q)$ is tangent to the leaf through Q of the Levi foliation, and $d\tilde{h}(\bar{L})(Q) = \bar{L}\tilde{h}(Q) = \bar{L}h(Q) = \alpha(\bar{L})(Q)$. Therefore, setting $h_\varepsilon := \tilde{h}$ on a neighborhood $U_\varepsilon = U$ of K gives a family that satisfies the requirements of Proposition 5.13. ☐

Remarks. (i) By Lemma 5.15, the class $[\alpha|_{\mathcal{L}}]$ on each leaf is intrinsic. This only reflects the fact that the leaf is a complex submanifold of the boundary, but not that it is a leaf in the Levi foliation. One could reasonably expect this class to also be linked to the foliation. This is indeed the case: $[\alpha|_{\mathcal{L}}]$ coincides with the infinitesimal holonomy of the leaf ([59], Example 2.3.15; [289], Remark 2). A detailed explanation of this connection may also be found in [7], Section 3.6; there, a thorough discussion of $[\alpha|_M]$, $[\alpha|_{\mathcal{L}}]$, etc., is provided from the point of view of foliation theory.

(ii) The assumption in Corollary 5.19 that the Levi foliation of K is part of the Levi foliation of a bigger Levi flat hypersurface is the strongest possible in terms of imposing control on the boundary behavior of the leaveson K. One can get results under weaker assumptions, especially in $\mathbb{C}^2$. This is done in [289], Theorem 1. The authors also show that the two conditions in Lemma 5.18 are equivalent to a third one in terms of the flow generated by the transverse field T ([289], Proposition 2). This equivalence is potentially of interest because the flow exists whether or not there is a foliation in the boundary. Using ideas from [290], [291], the authors then derive a characterization of when $dh|_{\mathcal{L}} = \alpha|_{\mathcal{L}}$ is solvable on K in terms of a homological condition on foliation currents associated to the flow generated by the transverse field T ([289], Theorem 3).

5.12 Worm domains

The results so far show that (exact) global regularity for the $\bar{\partial}$-Neumann operator holds on large classes of domains. However, it does not hold on all smooth bounded pseudoconvex domains. The known counterexamples are the so called worm domains of

Diederich and Fornaess. This family of domains was introduced in [105] as examples of smooth bounded pseudoconvex domains whose closure does not admit a Stein neighborhood basis (for suitable values of the parameter). But, as the authors remark, 'once it was at the disposal, it seemed to be useful to test also several other questions about pseudoconvex domains ...'. This turned out to be the case also for global regularity.

To define the worm domains (following [81]), fix $r > 0$ and a function φ_r with the following properties: (i) φ_r is even and convex, $\varphi_r \geq 0$, (ii) $\varphi_r(x) = 0 \Leftrightarrow x \in [-r, r]$, (iii) there exists $a > 0$ such that $\varphi_r(x) > 1$ if $|x| > a$, (iv) $\varphi_r'(x) \neq 0$ if $\varphi_r(x) = 1$. Then we set

$$\Omega_r = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1 + e^{i \log|z_2|^2}|^2 < 1 - \varphi_r(\log|z_2|^2)\}. \tag{5.103}$$

Note that Ω_r has rotational symmetry in the second variable and so is a Hartogs domain. The intersection with $z_2 = \text{const.}$ is a disc centered at $-e^{i \log|z_2|^2}$ and radius $\sqrt{1 - \varphi_r(\log|z_2|^2)}$. This radius is 1 for $e^{-r/2} \leq |z_2| \leq e^{r/2}$, then decreases to 0 as $|z_2|$ increases past $e^{r/2}$ or decreases below $e^{-r/2}$ (note that (iii) guarantees that $|z_2| > 0$). The latter parts of the domain are usually referred to as the 'caps'. Note that we can choose a as close to r as we wish, so that these caps are very 'flat'. In the three dimensional picture, with coordinates $(z_1 = x_1 + iy_1, |z_2|)$, the discs wind around the $|z_2|$-axis, so that this image of the domain looks somewhat like a worm. The parameter r determines the total amount of winding around the $|z_2|$-axis (when the caps are flat, so that their contribution to the winding is negligible): the center of the disc winds by $2r$ radians.

Lemma 5.20. *Ω_r is a smooth bounded pseudoconvex domain. It is strictly pseudoconvex except at points of the annulus $A := \{(0, z_2) \in \mathbb{C}^2 \mid e^{-r/2} \leq |z_2| \leq e^{r/2}\}$. The class $[\alpha|_A]$ is nonzero; in particular, Ω_r does not admit a defining function that is plurisubharmonic at the boundary. Nonetheless, near every boundary point, there is a local plurisubharmonic defining function.*

Proof. It is clear that Ω_r is bounded. To see that it is smooth, we check the gradient of a defining function on the boundary. Taking

$$\begin{aligned} \rho(z_1, z_2) &= \left(z_1 + e^{i \log|z_2|^2}\right)\left(\overline{z_1} + e^{-i \log|z_2|^2}\right) + \varphi_r(\log|z_2|^2) - 1 \\ &= |z_1|^2 + 2\operatorname{Re}\left(z_1 e^{-i \log|z_2|^2}\right) + \varphi_r(\log|z_2|^2), \end{aligned} \tag{5.104}$$

we have

$$\frac{\partial \rho}{\partial z_1} = \overline{z_1} + e^{-i \log|z_2|^2}, \tag{5.105}$$

and

$$\begin{aligned} \frac{\partial \rho}{\partial z_2} &= i e^{i \log|z_2|^2} \frac{1}{z_2}\left(\overline{z_1} + e^{-i \log|z_2|^2}\right) \\ &\quad - \left(z_1 + e^{i \log|z_2|^2}\right) e^{-i \log|z_2|^2} \frac{i}{z_2} + \varphi_r'\left(\log|z_2|^2\right) \frac{1}{z_2}. \end{aligned} \tag{5.106}$$

At a boundary point where $\partial\rho/\partial z_1 = 0$, we must have $\varphi_r(\log|z_2|^2) = 1$ (by (5.104)), so that $\partial\rho/\partial z_2 \neq 0$ (by (5.106) and property (iv)). Thus the gradient of ρ does not vanish on $b\Omega_r$, and Ω_r is smooth.

Near a boundary point, there is a smooth branch of $\arg(z_2)$ and an associated holomorphic branch of $\log(z_2)$ (since no points with $z_2 = 0$ are in the boundary), and we can take

$$\begin{aligned}\tilde\rho(z_1, z_2) &= e^{\arg(z_2^2)}\rho(z_1, z_2)\\ &= |z_1|^2 e^{\arg(z_2^2)} + 2\operatorname{Re}\left(z_1 e^{-i\log(z_2^2)}\right) + \varphi_r\left(\log|z_2|^2\right)e^{\arg(z_2^2)}\end{aligned} \tag{5.107}$$

as a local defining function. The first and the second term on the right-hand side of (5.107) are, respectively, the modulus of $z_1^2 e^{-i\log(z_2^2)}$, and twice the real part of $z_1 e^{-i\log(z_2^2)}$, and thus are plurisubharmonic. For the third term, we have (by computation)

$$\frac{\partial^2}{\partial z_2 \partial\bar z_2}\varphi_r(\log|z_2|^2)e^{\arg(z_2^2)} = \frac{\left(\varphi_r''(\log|z_2|^2) + \varphi_r(\log|z_2|^2)\right)e^{\arg(z_2^2)}}{|z_2|^2} \geq 0 \tag{5.108}$$

(φ_r is convex and ≥ 0). This shows that Ω_r admits local plurisubharmonic defining functions near every boundary point; in particular, Ω_r is pseudoconvex. Computing the Hessian of $|z_1|^2 e^{\arg(z_2^2)} = \left|z_1 e^{-i\log z_2}\right|^2$ (the first term on the right-hand side of (5.107)), gives

$$\begin{aligned}\sum_{j,k=1}^{2}\frac{\partial^2\left(|z_1|^2 e^{\arg(z_2^2)}\right)}{\partial z_j\partial\bar z_k}w_j\overline{w_k} &= \left|\sum_{j=1}^{2}\frac{\partial\left(z_1 e^{-i\log z_2}\right)}{\partial z_j}w_j\right|^2\\ &= \left|e^{-i\log z_2}w_1 - \frac{iz_1}{z_2}e^{-i\log z_2}w_2\right|^2\\ &= \left|e^{-i\log z_2}\right|^2\left|w_1 - i\frac{z_1}{z_2}w_2\right|^2.\end{aligned} \tag{5.109}$$

At a boundary point in one of the caps (i.e., where $|z_2| < e^{-r/2}$ or $|z_2| > e^{r/2}$), (5.108) is strictly positive. Therefore, the Hessian of $\tilde\rho$, applied to a vector $(w_1, w_2) \neq (0,0)$, is strictly positive as long as $w_2 \neq 0$. If $w_2 = 0$, it is strictly positive in view of (5.109). (We also use that each term in (5.107) has nonnegative Hessian.) In particular, Ω_r is strictly pseudoconvex at points of the caps. Now assume that $e^{-r/2} \leq |z_2| \leq e^{r/2}$. Then only the first term on the right-hand side of (5.107), contributes to the Hessian of $\tilde\rho$; this Hessian is therefore given by (5.109). Inserting the complex tangent vector $(-\partial\tilde\rho/\partial z_2, \partial\tilde\rho/\partial z_1)$ into (5.109) shows that the Levi form vanishes precisely when $z_1 = 0$, i.e., on A.

To check that $[\alpha|_A] \neq 0$ also amounts to a computation. On A, the normal L_2 is $e^{i\log|z_2|^2}(\partial/\partial z_1)$. We may take L to be $\partial/\partial z_2$. Then

$$[L_2, \partial/\partial\bar z_2] = (-i/\overline{z_2})e^{i\log|z_2|^2}(\partial/\partial z_1).$$

Therefore

$$\alpha\left(\frac{\partial}{\partial\bar{z}_2}\right) = \partial\rho\left(\left[L_2, \frac{\partial}{\partial\bar{z}_2}\right]\right) = \frac{-i}{\overline{z_2}}. \tag{5.110}$$

Thus (recall that α is real)

$$\alpha|_A = \frac{i}{z_2}dz_2 - \frac{i}{\overline{z_2}}d\bar{z}_2. \tag{5.111}$$

When $e^{-r/2} < a < e^{r/2}$, the circle $|z_2| = a$ is a concentric circle in A, and $[\alpha|_A] \neq 0$ will follow from $\int_{|z_2|=a}\alpha \neq 0$. But

$$\int_{|z_2|=a}\alpha = \int_{|z_2|=a}\left(\frac{i}{z_2}dz_2 - \frac{i}{\overline{z_2}}d\bar{z}_2\right) = 2\,\mathrm{Re}\left(i\int_{|z_2|=a}\frac{dz_2}{z_2}\right) = -4\pi. \tag{5.112}$$

We have seen that when a domain admits a defining function that is plurisubharmonic at the boundary, then the form α resulting from choosing this defining function is zero on the null space of the Levi form, in particular, it vanishes when restricted to a complex submanifold in the boundary (see the proof of Corollary 5.11). Then its cohomology class on this submanifold is zero. Since this class is independent of the choice of defining function (Lemma 5.15), $[\alpha|_A] \neq 0$ as above implies that Ω_r does not admit a defining function that is plurisubharmonic at the boundary. This completes the proof of Lemma 5.20. □

Once worm domains are at hand, it is not hard to see that their closure does not admit a Stein neighborhood basis when the amount of winding is not too small. Indeed, when $r \geq \pi$, any open set that contains the closure of Ω_r must contain a neighborhood of the compact set $A \cup \{(z_1, z_2) \mid |z_1 + e^{-ir}|^2 \leq 1, |z_2| = e^{-r/2}\} \cup \{(z_1, z_2) \mid |z_1 + e^{-i(r+2\pi)}|^2 \leq 1, |z_2| = e^{(-r+2\pi)/2}\}$. If the open set is pseudoconvex, it must then also contain the 'filled in' part $\{|z_1 + e^{-ir}|^2 \leq 1\} \times \{e^{-r/2} \leq |z_2| \leq e^{(-r+2\pi)/2}\}$. For this and further interesting properties of the worm domains, we refer the reader to [105], [130], [214], [43], [208]. For the role that the class $[\alpha|_A]$ plays in this context, the reader should consult [17]. For us, the main interest in the worm domains stems from the fact that on them, global regularity fails.

The first indication that things can go wrong with respect to estimates even on smooth domains came in [9], where Barrett exhibited smooth bounded Hartogs domains in $\mathbb{C}^2$ whose Bergman projection P_0 does not preserve smoothness up to the boundary. These domains were however not pseudoconvex. A few years later, Kiselman ([183]) discovered that on worm domains with the caps removed, one can exploit a certain resonance phenomenon to again show that P_0 does not preserve smoothness up to the boundary. Of course, removing the caps means that the resulting domain is not smooth. In [11], Barrett proved that on Ω_r, P_0 does not preserves the Sobolev spaces $W^s(\Omega_r)$ when $s \geq \pi/2r$. This left open the possibility that P_0 preserves $C^\infty(\overline{\Omega_r})$ (with loss of derivatives, that is, for each $s > 0$ there exists $k(s)$ such that P_0 is continuous from $W^{k(s)}(\Omega_r)$ to $W^s(\Omega_r)$). This question is very important from the point of view of several complex variables, since this regularity property is good enough to imply smooth extension to the boundary of biholomorphic and proper holomorphic mappings ([32], [23], [30], [108]). The issue was resolved by Christ ([83], see also [84], [85]).

Theorem 5.21. *If Ω_r is one of the worm domains above, then both N_1 and P_0 fail to map $C^{\infty}(\bar{\Omega})$ to itself (at the level of $(0,1)$-forms and functions, respectively).*

We refer the reader to [83] for the proof. The proof uses Barrett's result from [11] in a rather unexpected way. Namely, Christ shows that for most $s > 0$, N_1 on Ω_r satisfies a priori estimates in $W^s_{(0,1)}(\Omega)$: there is a constant C_s such that when both u and $N_1 u$ are in $C^{\infty}_{(0,1)}(\bar{\Omega})$, then $\|N_1 u\|_s \leq C_s\|u\|_s$ holds. If N_1 were to preserve $C^{\infty}_{(0,1)}(\bar{\Omega})$, this would imply Sobolev estimates. Indeed, if $u \in C^{\infty}_{(0,1)}(\overline{\Omega_r})$, then $N_1 u \in C^{\infty}_{(0,1)}(\overline{\Omega_r})$, and the a priori estimate holds:

$$\|N_1 u\|_s \leq C_s\|u\|_s. \tag{5.113}$$

Because $C^{\infty}_{(0,1)}(\overline{\Omega_r})$ is dense in $W^s_{(0,1)}(\Omega_r)$ and N_1 is continuous in $\mathcal{L}^2_{(0,1)}(\Omega_r)$, estimate (5.113) carries over to all of $W^s_{(0,1)}(\Omega_r)$. But then Theorem 5.5 implies Sobolev estimates for P_0 as well, contradicting [11]. This contradiction establishes the result.

Remarks. (i) Christ's analysis shows in particular that one has to pay attention to the issue of a priori estimates versus genuine estimates; a priori estimates can not always be converted into genuine estimates. For this phenomenon in the context of the Bergman projection, see [49].

(ii) At points of A, the normal is invariant under rotations in the z_2-variable. If r is small it only winds little ($2r$ radians) when passing from the inner boundary of A to the outer boundary. Therefore, it can be well approximated on A by a constant field. More precisely, a constant field satisfies (i) in Theorem 5.7 for $\varepsilon \approx r$ on $K = A$ (it satisfies (ii) for all ε). Since size estimates on derivatives of the defining function do not depend on r, this allows to prove Sobolev estimates up to a certain level. This is made precise in [49], Proposition 1, where the authors show that if $k \in \mathbb{N}$ and $k < \pi/4r$, then P_0 satisfies estimates in $W^k(\Omega_r)$. Note that this level is half the level above which estimates are known to fail (by [11]). It is not known what happens at the levels in between. If, however, weighted projections are allowed, say with weights that are rotationally symmetric in the second variable, the discrepancy disappears almost completely. There are weights where the threshold of failure is as close to $\pi/4r$ as we wish, while the methods of [49] still apply and give estimates at levels below $\pi/4r$. For more details, see [11], Remark 4.

(iii) From Lemma 5.20 we know that the domains Ω_r admit local plurisubharmonic defining functions. The proofs of Corollary 5.11 and Theorem 5.9 show that there are open sets $U_1, \dots, U_M$ that cover A and families of vector fields $\{X_{k,\varepsilon}\}$ as in Theorem 5.7, except that $\{X_{k,\varepsilon}\}$ is only defined on U_k. (We may in fact take $M = 2$.) Nonetheless, global regularity fails. This shows that the assumption in Theorem 5.7 that the families $\{X_{k,\varepsilon}\}$ be defined in a neighborhood U_ε of K and satisfy all the conditions, except transversality, there (rather than just in $U_\varepsilon \cap U_k$), is needed.

5.13 A unified approach to global regularity

We have seen two distinct avenues to global regularity in the $\bar{\partial}$-Neumann problem: one is via compactness (Theorem 4.6), the other via what we may call the 'vector field method' (Theorem 5.7). We now show how these two approaches can be combined. We treat in detail a version that corresponds to the case $M = 1$ in Theorem 5.7. The following theorem comes essentially from [287].

Theorem 5.22. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, ρ a defining function for Ω. Let $1 \leq q \leq n$. Assume that there is a constant C such that for all $\varepsilon > 0$ there is a function $h_\varepsilon \in C^\infty(\overline{\Omega})$ with*

$$|h_\varepsilon| \leq C; \quad |\mathrm{Im}(h_\varepsilon)| \leq \varepsilon, \tag{5.114}$$

and

$$\begin{aligned} &\left\| \sum_{|K|=q-1}' \left(\sum_{j,k=1}^{n} \frac{\partial^2 (e^{h_\varepsilon}\rho)}{\partial z_j \partial \bar{z}_k} \frac{\partial \rho}{\partial \bar{z}_j} \overline{u_{kK}} \right) d\bar{z}_K \right\|^2 \\ &\qquad \leq \varepsilon\left(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2\right) + C_\varepsilon \|u\|_{-1}^2 \end{aligned} \tag{5.115}$$

for all $u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*)$. Then the $\bar{\partial}$-Neumann operator N_q on $(0,q)$-forms is continuous on $W^s_{(0,q)}(\Omega)$ when $s \geq 0$.

Remarks. (i) As in Theorem 5.7, the particular form of the bound ε is not important; it suffices to have a bound of the form $g(\varepsilon)$ with $g(\varepsilon) \to 0$ when $\varepsilon \to 0^+$.

(ii) Whether or not a constant C and a family $\{h_\varepsilon\}$ exists satisfying (5.114) and (5.115) does not depend on the choice of defining function ρ: if $\tilde{\rho} = e^h \rho$, the family $\{h_\varepsilon - h\}$ works for $\tilde{\rho}$. The extra term arising from $\partial(e^h\rho)/\partial\bar{z}_j$ is dominated by $C_\varepsilon\|\rho u\|^2$; it can be estimated by the right-hand side of (5.115) in the same manner as the term $\|\sqrt{-\rho}u\|^2$ in (5.117) below (replacing Ω_ε there by $\Omega_{(\varepsilon/C_\varepsilon)^{1/2}}$).

(iii) Actually, Theorem 5.22 is stated in [287] only when the functions h_ε are real. In this case, $e^{h_\varepsilon}\rho$ is a defining function for Ω, denoted ρ_ε in Theorem 1 in [287]. However, the arguments in [287] work in the slightly more general situation of Theorem 5.22. We also take this opportunity to correct an error in [286], [287] that is relevant in this context. It is claimed in [287], Proposition 1 (and, as a result, in Proposition 3.6 in [286]), that when there is a family of vector fields X_ε as in Theorem 5.9, then a family $\{h_\varepsilon\}$ of real-valued functions exists that satisfies the assumptions in Theorem 5.22 (rather than one where the imaginary part is allowed to be $O(\varepsilon)$). We give a corrected version in Proposition 5.26 below. The argument in the proof of Proposition 1 in [287] works only in the case where the family X_ε satisfies the more stringent condition that $X_\varepsilon\rho$ is real in the neighborhood U_ε of K. This remark also applies to the main theorem in [288] (see also the remark following (5.85)). On the other hand, in the cases where the vector field method applies (Corollaries 5.8, 5.11, 5.12, 5.17, 5.16, 5.19), one actually does have $X_\varepsilon\rho$ real in the neighborhood U_ε of K. This is obvious, except perhaps for Corollary 5.16, where it is noted in Remark (iii) following its proof.

(iv) The left-hand side of (5.115) is stated in terms of a global $\mathcal{L}^2$-norm. Nonetheless, all the action is still near K, the set of points of infinite type. If U_ε is a neighborhood of K, then the portion of the $\mathcal{L}^2$-norm in (5.115) on $\Omega \setminus U_\varepsilon$ can be handled by the pseudolocal subelliptic estimates (see (3.75) and the discussion following its statement; in the strictly pseudoconvex case, see Theorem 3.6) to obtain the upper bound given by the right-hand side of (5.115) (again in the sense of Remark (i)). Details of this argument are in the proof of Proposition 5.26 below. Moreover, it suffices to control the contribution of components of u in directions that are close to the null space of the Levi form, in the following sense. The integrand on the left-hand side of (5.115) is (pointwise) dominated by $C_\varepsilon|u|^2$. But, (very) roughly speaking, at points where $u(z)$ stays away from the nullspace of the Levi from, the Levi form controls $|u|^2$. Consequently, the portion of the integral over the set of these points is dominated by $C_\varepsilon \int_\Omega \sum_{j,k=1}^n (\partial^2\rho/\partial z_j \partial\bar{z}_k) u_j \overline{u_k}$ (assume $q = 1$ for simplicity). This integral in turn can be estimated as required by the right-hand side of (5.115), see (5.117)–(5.122) below. In other words, the estimate for this portion of the norm on the left-hand side of (5.115) always holds. The arguments in (5.117)–(5.122) also show how to account for the fact that u need not be tangential to the level sets of ρ, and that these level sets need not be pseudoconvex.

We postpone the proof of Theorem 5.22 and first discuss various aspects of the theorem. In doing so, we follow [287].

It is obvious that when N_q is compact, the assumption in Theorem 5.22 is satisfied. Indeed, it suffices to set $h_\varepsilon \equiv 0$ for all $\varepsilon > 0$. Then the left-hand side is bounded by $C\|u\|^2$, which is bounded by the right-hand side of (5.115) (this is the compactness estimate in Proposition 4.2).

We will show below (Proposition 5.26) that when the assumptions of Theorem 5.7 are satisfied, then so are the assumptions in Theorem 5.22 (at all form levels q). But it is worthwhile to discuss separately the special case when the domain admits a defining function ρ that is plurisubharmonic at the boundary. The simple solution of taking $h_\varepsilon \equiv 0$ for all $\varepsilon > 0$ also works in this case. We claim that then (5.115) holds for $q = 1$ (and hence for all q, in view of Lemma 5.23 below). Take $h_\varepsilon \equiv 0$. There is a constant C such that the complex Hessian of ρ is bounded from below by $C\rho$ near $b\Omega$. This means that subtracting the form $C\rho|u|^2$ results in a form that is positive semidefinite near $b\Omega$ (note that ρ is negative on Ω). Applying the Cauchy–Schwarz inequality to this form pointwise gives, near $b\Omega$,

$$\begin{aligned} \left| \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k} \frac{\partial\rho}{\partial\bar{z}_j} \overline{u_k} \right|^2 &\lesssim \left| \sum_{j,k=1}^n \left(\frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k} - C\rho\delta_{j,k} \right) \frac{\partial\rho}{\partial\bar{z}_j} \overline{u_k} \right|^2 + C^2\rho^2|u|^2 \\ &\lesssim \sum_{j,k=1}^n \left(\frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k} - C\rho\delta_{j,k} \right) u_j \overline{u_k} + C^2\rho^2|u|^2. \end{aligned} \tag{5.116}$$

Integrating over Ω shows that the left-hand side of (5.115) (with $h_\varepsilon \equiv 0$) is bounded from above by

$$\left| \int_{\Omega} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} u_j \overline{u_k} \right| + \|\sqrt{(-\rho)}u\|^2 + \|u\|_V^2, \tag{5.117}$$

where V is a big enough relatively compact subdomain of Ω (we may assume that $\rho^2 \leq -\rho$ on $\Omega \setminus V$). $\|\sqrt{(-\rho)}u\|^2 \leq \varepsilon\|u\|^2 + C\|u\|^2_{\Omega_\varepsilon} \lesssim \varepsilon\big(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2\big) + \|u\|^2_{\Omega_\varepsilon}$, where Ω_ε denotes the relatively compact subdomain of Ω consisting of the points whose distance to the boundary exceeds ε. Here, the last inequality is from Proposition 2.7. Because of interior elliptic regularity of $\bar{\partial} \oplus \bar{\partial}^*$ (see (2.85)) and interpolation of Sobolev norms (i.e., the estimate $\|u\|^2 \leq \varepsilon\|u\|_1^2 + C_\varepsilon\|u\|_{-1}^2$), the compactly supported terms are easily seen to be dominated in the manner required by the right-hand side of (5.115). To estimate the first term in (5.117), we split u into its normal and tangential parts: near $b\Omega$, $u = u_{\text{Norm}} + u_{\text{Tan}}$. If φ is a cutoff function whose support is contained in a small enough neighborhood of $b\Omega$, and which is equal to one near $b\Omega$, we can write $u = (1-\varphi)u + \varphi u_{\text{Norm}} + \varphi u_{\text{Tan}}$. The first term in (5.117) is then dominated (with a constant that does not depend on ε) by

$$\left| \int_{\Omega} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} (\varphi u_{\text{Tan}})_j \overline{(\varphi u_{\text{Tan}})_k} \right| + \varepsilon\|\varphi u_{\text{Tan}}\|^2 + C_\varepsilon\|\varphi u_{\text{Norm}}\|^2 + \|(1-\varphi)u\|^2. \tag{5.118}$$

We have used the inequality $|\varphi u_{\text{Tan}}||\varphi u_{\text{Norm}}| \leq \varepsilon|\varphi u_{\text{Tan}}|^2 + C_\varepsilon|\varphi u_{\text{Norm}}|^2$. The last term in (5.118) is compactly supported; it can be estimated in the same way the compactly supported terms above were estimated. For $C_\varepsilon\|\varphi u_{\text{Norm}}\|^2$, we have

$$\begin{aligned} C_\varepsilon\|\varphi u_{\text{Norm}}\|^2 &\leq \varepsilon\|\varphi u_{\text{Norm}}\|_1^2 + \tilde{C}_\varepsilon\|\varphi u_{\text{Norm}}\|_{-1}^2 \\ &\lesssim \varepsilon\big(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2\big) + \tilde{C}_\varepsilon\|u\|_{-1}^2. \end{aligned} \tag{5.119}$$

The first inequality follows from interpolation of Sobolev norms, the second from Lemma 2.12. Since $\|\varphi u_{\text{Tan}}\|^2 \leq \|u\|^2 \lesssim \|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2$ (Proposition 2.7), it only remains to estimate the first term in (5.118). We split the integral over Ω into one over $\Omega \setminus \Omega_\varepsilon$ plus one over Ω_ε. The latter is compactly supported and can be estimated as the other compactly supported terms above. For the former, we use Fubini's theorem and integrate first over the level sets $b\Omega_\delta$ of ρ, $0 \leq \delta \leq \varepsilon$, and then over δ. This gives

$$\begin{aligned} &\left| \int_{\Omega\setminus\Omega_\varepsilon} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} (\varphi u_{\text{Tan}})_j \overline{(\varphi u_{\text{Tan}})_k} \right| \\ &\leq \int_0^\varepsilon \left| \int_{b\Omega_\delta} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} (\varphi u_{\text{Tan}})_j \overline{(\varphi u_{\text{Tan}})_k} \frac{d\sigma_\delta}{|\nabla\rho|} \right| d\delta \end{aligned} \tag{5.120}$$

(see e.g. [259], Chapter III, Lemma 3.4 for a convenient expression for $d\sigma_\delta$ that shows that $(d\sigma_\delta/|\nabla\rho|)d\delta$ corresponds to dV). Because φu_{Tan} is in the domain of $\bar{\partial}^*$ on Ω_δ,

$0 \le \delta \le \varepsilon$, we can apply Proposition 2.4 on Ω_δ, with $a \equiv 1$ and weight 0. As a result, the right-hand side of (5.120) is dominated by

$$\begin{aligned}\int_0^\varepsilon &\left(\|\bar\partial(\varphi u_{\mathrm{Tan}})\|^2_{\Omega_\delta} + \|\bar\partial^*(\varphi u_{\mathrm{Tan}})\|^2_{\Omega_\delta} + \sum_{j,k=1}^n \left\| \frac{\partial(\varphi u_{\mathrm{Tan}})_j}{\partial \bar z_k} \right\|^2_{\Omega_\delta} \right) d\delta \\ &\le \varepsilon \left(\|\bar\partial(\varphi u_{\mathrm{Tan}})\|^2 + \|\bar\partial^*(\varphi u_{\mathrm{Tan}})\|^2 + \sum_{j,k=1}^n \left\| \frac{\partial(\varphi u_{\mathrm{Tan}})_j}{\partial \bar z_k} \right\|^2 \right) \\ &\lesssim \varepsilon \left(\|\bar\partial(\varphi u_{\mathrm{Tan}})\|^2 + \|\bar\partial^*(\varphi u_{\mathrm{Tan}})\|^2 \right).\end{aligned} \tag{5.121}$$

The last inequality is again from Proposition 2.4 (Ω is pseudoconvex, while the Ω_δ need not be). Finally, by Lemma 2.12, Proposition 2.7, and the decomposition $u = (1-\varphi)u + \varphi u_{\mathrm{Tan}} + \varphi u_{\mathrm{Norm}}$,

$$\begin{aligned}\|\bar\partial(\varphi u_{\mathrm{Tan}})\|^2 + \|\bar\partial^*(\varphi u_{\mathrm{Tan}})\|^2 &\lesssim \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + \|u\|^2 + \|\varphi u_{\mathrm{Norm}}\|_1^2 \\ &\lesssim \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2.\end{aligned} \tag{5.122}$$

We have shown: if ρ is a defining function for Ω that is plurisubharmonic at points of the boundary, then (5.115) holds with $h_\varepsilon \equiv 0$ for $\varepsilon > 0$.

Remarks. (i) It was convenient to split u into its normal and tangential parts (near $b\Omega$) in order to apply Proposition 2.4, which is formulated only for $u \in \mathrm{dom}(\bar\partial^*)$. If u is not assumed in $\mathrm{dom}(\bar\partial^*)$, there are additional terms. This results in additional (but benign) terms for $\int_{b\Omega_\delta} (\partial^2\rho/\partial z_j \partial \bar z_k) u_j \overline{u_k}\, d\sigma_\delta$; compare formula (1) in [293].

(ii) In view of Lemma 4.3, (5.115) holds with a fixed real valued function $h_\varepsilon = h$ if and only if there is a defining function $\hat\rho = e^h \rho$ such that the operator

$$\begin{aligned}B_{\hat\rho}(u) &:= {\sum_{|K|=q-1}}' \left(\sum_{j,k=1}^n \frac{\partial^2(\hat\rho)}{\partial z_j \partial \bar z_k} \frac{\partial \hat\rho}{\partial \bar z_j} \overline{u_{kK}} \right) d\bar z_K, \\ &u \in \mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*) \subset \mathcal{L}^2_{(0,q)}(\Omega),\end{aligned} \tag{5.123}$$

is compact as an operator from $\mathrm{dom}(\bar\partial) \cap \mathrm{dom}(\bar\partial^*) \subset \mathcal{L}^2_{(0,q)}(\Omega)$ (with the graph norm) to $\mathcal{L}^2_{(0,q-1)}(\Omega)$. This is a compactness property considerably weaker than compactness of the $\bar\partial$-Neumann operator that still implies global regularity. That this compactness property is much weaker than compactness of the $\bar\partial$-Neumann operator can be seen by considering convex domains. They always admit a defining function ρ that is plurisubharmonic at the boundary (see the discussion following the statement of Corollary 5.11), so that B_ρ is always compact. The $\bar\partial$-Neumann operator, on the other hand, is compact (if and) only if the boundary contains no q-dimensional varieties (Theorem 4.26).

The sufficient conditions for global regularity in the $\bar\partial$-Neumann problem in Theorem 5.22 percolate up the $\bar\partial$-complex. It would be interesting to know whether global regularity itself does so also (compactness and subellipticity do, see Proposition 4.5).

Lemma 5.23. *Suppose the assumptions in Theorem 5.22 are satisfied for q-forms, where $1 \le q \le n-1$. Then they are satisfied for $(q+1)$-forms. Moreover, up to rescaling, the same family of functions $\{h_\varepsilon\}$ may be taken.*

Proof. The proof is analogous to the proof of Proposition 4.5. Let $u \in C^\infty_{(0,q+1)}(\overline{\Omega}) \cap \operatorname{dom}(\bar\partial^*)$. For $1 \le k \le n$, define the $(0,q)$-forms $v_k = \sum'_{|K|=q} u_{kK}\, d\bar z_K$. Then $v_k \in \operatorname{dom}(\bar\partial^*)$ and (see (4.15))

$$\bar\partial^* v_k = -\sum_{|S|=q-1}{}' (\bar\partial^* u)_{kS}\, d\bar z_S. \tag{5.124}$$

Now observe that

$$\sum_{|K|=q}{}' \int_\Omega \left| \sum_{j,k=1}^{n} \frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j \partial \bar z_k} \frac{\partial \rho}{\partial \bar z_j} \overline{u_{kK}} \right|^2 = \sum_{(m,\widehat{K})}{}' \int_\Omega \left| \sum_{j,k=1}^{n} \frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j \partial \bar z_k} \frac{\partial \rho}{\partial \bar z_j} \overline{u_{km\widehat{K}}} \right|^2, \tag{5.125}$$

where the outer sum on the right-hand side is only over those pairs $(m, \widehat{K})$ such that $(m, \widehat{K})$ is an increasing q-tuple. This sum is thus bounded from above by the sum over all m and all increasing $(q-1)$-tuples. Also replacing $u_{km\widehat{K}}$ by $-u_{mk\widehat{K}} = -(v_m)_{k\widehat{K}}$ and using (5.115) for q-forms, we find that the right-hand side of (5.125) is dominated by

$$\begin{aligned} &\sum_{m=1}^{n} \sum_{|\widehat{K}|=q-1}{}' \int_\Omega \left| \sum_{j,k=1}^{n} \frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j \partial \bar z_k} \frac{\partial \rho}{\partial \bar z_j} \overline{(v_m)_{k\widehat{K}}} \right|^2 \\ &\qquad \le \varepsilon \sum_{m=1}^{n} \left(\|\bar\partial v_m\|^2 + \|\bar\partial^* v_m\|^2 \right) + C_\varepsilon \sum_{m=1}^{n} \|v_m\|_{-1}^2. \end{aligned} \tag{5.126}$$

$\bar\partial v_m$ involves only barred derivatives of the coefficients of v_m, hence of the coefficients of u. Their $\mathcal{L}^2$-norms squared are bounded by $\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$ (Proposition 2.4). Similarly, $\|\bar\partial^* v_m\|^2$ is dominated by $\|\bar\partial^* u\|^2$, in view of (5.124). Also, $\|v\|_{-1}^2$ is dominated by $\|u\|_{-1}^2$ (from the definition of v_m). Therefore, combining (5.125) and (5.126) gives (5.115) for u, completing the proof of Lemma 5.23. □

There is a useful reformulation of condition (5.115) which will be used in the proof of Corollary 5.25 below. We set (departing slightly from (4.32))

$$H_{\rho,q}(u,\bar u) = \sum_{|K|=q-1}{}' \left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \bar z_k} u_{jK} \overline{u_{kK}} \right). \tag{5.127}$$

Lemma 5.24. *Assumptions as in Theorem 5.22, ρ is a defining function for Ω. Then, modulo rescaling, a family of functions $\{h_\varepsilon\}$ satisfies (5.114) and (5.115) if and only if* *it satisfies* (5.114) *and*

$$\sup_{\substack{\beta\in C^\infty_{(0,q-1)}(\overline{\Omega})\\ \|\beta\|\le 1}} \left\{\left|\int_\Omega H_{e^{h_\varepsilon}\rho,q}\big(\bar\partial\rho\wedge\beta,\bar u\big)\right|^2\right\} \le \varepsilon\big(\|\bar\partial u\|^2+\|\bar\partial^* u\|^2\big)+\widetilde{C_\varepsilon}\|u\|^2_{-1}. \tag{5.128}$$

Proof. If $q=1$, this equivalence is clear from the variational characterization of the norm. If $q>1$, the argument needs a little more care. For $\beta=\sum'_{|K|=q-1} b_K d\bar z_K\in C^\infty_{(0,q-1)}(\overline{\Omega})$, denote by $\tilde\beta$ the form $\tilde\beta=\sum'_{|K|=q-1}\overline{b_K}d\bar z_K$. Then

$$\begin{aligned} &H_{e^{h_\varepsilon}\rho,q}(\bar\partial\rho\wedge\tilde\beta,\bar u)\\ &\quad=\sideset{}{'}\sum_{|K|=q-1}\sum_{j,k=1}^{n}\frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j\partial\bar z_k}\big(\bar\partial\rho\wedge\tilde\beta\big)_{jK}\overline{u_{kK}}\\ &\quad=\frac{1}{(q-1)!}\sum_{k_1,\dots,k_{q-1}=1}^{n}\sum_{j,k=1}^{n}\frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j\partial\bar z_k}\big(\bar\partial\rho\wedge\tilde\beta\big)_{jk_1\dots k_{q-1}}\overline{u_{kk_1\dots k_{q-1}}}. \end{aligned} \tag{5.129}$$

The factor $1/(q-1)!$ arises as usual because each increasing $(q-1)$-tuple K occurs precisely $(q-1)!$ times as the ordered tuple of some $k_1\dots k_{q-1}$ in the unrestricted sum. Now

$$\begin{aligned}\big(\bar\partial\rho\wedge\tilde\beta\big)_{jk_1\dots k_{q-1}}&=\big(\bar\partial\rho\big)_j\,\tilde\beta_{k_1\dots k_{q-1}}+\text{‘error terms’}\\ &=\frac{\partial\rho}{\partial\bar z_j}\,\overline{b_{k_1\dots k_{q-1}}}+\text{‘error terms’}.\end{aligned} \tag{5.130}$$

The ‘error terms’ have the form (up to sign) $(\partial\rho/\partial\bar z_{k_s})\overline{b_{k_1\dots j\dots k_{q-1}}}$, $1\le s\le(q-1)$. The contribution from the first term on the right-hand side of (5.130) to (5.129) equals the (pointwise) inner product of the form on the left-hand side of (5.115) with β. The contribution from an error term, after summing over k_s, will be a smooth multiple (depending on ε) of $\overline{b_{k_1\dots k_{q-1}}}\sum_{k_s=1}^{n}(\partial\rho/\partial\bar z_{k_s})\,\overline{u_{kk_1\dots k_s\dots k_{q-1}}}$. The part in the sum is the conjugate of a coefficient of the normal component u_{norm} of u (see (2.86)), as $\overline{u_{kk_1\dots k_s\dots k_{q-1}}}$ equals plus or minus $\overline{u_{k_sk_1\dots k\dots k_{q-1}}}$. Therefore,

$$\begin{aligned}&\left(\sideset{}{'}\sum_{|K|=q-1}\left(\sum_{j,k=1}^{n}\frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j\partial\bar z_k}\frac{\partial\rho}{\partial\bar z_j}\overline{u_{kK}}\right)d\bar z_K,\ \beta\right)\\ &\quad=\int_\Omega H_{e^{h_\varepsilon}\rho,q}(\bar\partial\rho\wedge\tilde\beta,u)\,dV+\sideset{}{'}\sum_{|J|,|K|=q-1}\int_\Omega g_{\varepsilon,J,K}\overline{\beta_J(u_{\text{norm}})_K}\,dV,\end{aligned} \tag{5.131}$$

where the $g_{\varepsilon,J,K}$ are functions in $C^\infty(\overline{\Omega})$ that do not depend on β. For the terms in the sum on the right-hand side of (5.131), we have the estimate

$$
\begin{aligned}
\left|\int_\Omega g_{\varepsilon,J,K}\overline{\beta_J(u_{\mathrm{norm}})_K}\,dV\right| &\le C_\varepsilon\|\beta\|\|u_{\mathrm{norm}}\| \\
&\le \|\beta\|\left(\varepsilon\|u_{\mathrm{norm}}\|_1 + C_\varepsilon\|u_{\mathrm{norm}}\|_{-1}\right) \\
&\le \|\beta\|\left(\varepsilon C\left(\|\bar\partial u\| + \|\bar\partial^* u\|\right) + C_\varepsilon\|u\|_{-1}\right),
\end{aligned}
\tag{5.132}
$$

for a constant C that does not depend on ε. The inequality in the second line of (5.132) comes from interpolation of Sobolev norms, the one in the third line from Lemma 2.12. As usual, C_ε is allowed to change in the course of the estimates. Using the variational characterization of the norm, taking suprema over β with $\|\beta\| \le 1$, and invoking (5.132) to estimate the second term on the right-hand side of (5.131) proves the lemma. □

Besides unifying Theorems 4.6 and 5.7 (and corollaries), Theorem 5.22 also covers situations when $q > 1$ where Theorem 5.7 does not apply (more precisely: not without substantial modifications which would ultimately lead to Theorem 5.22). Let Ω be a smooth bounded domain in $\mathbb{C}^n$, and q_0 a fixed integer, $1 \le q_0 \le n$. Assume that the Levi from has the property that the sum of any q_0 eigenvalues is nonnegative (note that this statement does not depend on the choice of defining function). When $q_0 = 1$, this just says that Ω is pseudoconvex. However, when $q_0 > 1$, such a domain need not be pseudoconvex. Nonetheless, the theory developed in Chapter 2 still goes through for forms of degree $q \ge q_0$. To see this, recall how pseudoconvexity was used in Theorem 2.9. What was needed was that the boundary term $\sum'_K \sum_{j,k=1}^n a(\partial^2\rho/\partial z_j\partial\bar z_k)u_{jK}\overline{u_{kK}}e^{-\varphi}\,d\sigma$ in (2.24) is nonnegative. When $q > 1$, it is not necessary for this that all eigenvalues of the Levi form be nonnegative. Indeed, in view of Lemma 4.7, it suffices that the sum of any q eigenvalues is nonnegative. As a consequence, Theorem 2.9 remains true for $q \ge q_0$ on domains where the Levi form has this property, as does Theorem 5.5. Likewise, the proof of Theorem 5.22 will show that it similarly remains valid (when $q \ge q_0$) when the assumption of pseudoconvexity is replaced by the assumption that the Levi form has the property that the sum of any q_0 eigenvalues is nonnegative. Assume now that Ω admits a defining function ρ whose (full) Hessian has the property that at each boundary point the sum of any q_0 eigenvalues is nonnegative. (Note that in view of Lemma 4.7, equivalence of (ii) and (iii), the Levi form then has this property as well.) This notion is a natural generalization to the case $q > 1$ of the notion of a defining function that is plurisubharmonic at the boundary (and so in particular is not independent of the defining function). What we said earlier when Ω admits a defining function that is plurisubharmonic at the boundary applies equally well here, with the modification that Lemma 5.24 is needed: by Lemma 4.7, $H_{\rho,q}$ is positive semidefinite at points of the boundary. After adding a form that is $O(\rho)$, one can again invoke the Cauchy–Schwarz inequality (while the pairing in (5.115) is not between q-forms) and then estimate the left-hand side of (5.128) in a manner that is completely analogous to the case when $q = 1$ discussed in detail earlier. One thus verifies (5.128); Lemma 5.24 then gives (5.115). Consequently, Theorem 5.22 applies, and we obtain the following result from [169].

Corollary 5.25. *Let Ω be a smooth bounded domain in $\mathbb{C}^n$, $1 \le q_0 \le n$. Assume that Ω admits a defining function with the property that the sum of any q_0 eigenvalues of its Hessian is nonnegative at each point of the boundary. Then the $\bar\partial$-Neumann operators N_q, $q_0 \le q \le n$, and the Bergman projections P_q, $q_0 - 1 \le q \le n$, are continuous in $W^s_{(0,q)}(\Omega)$ for $s \ge 0$.*

Remark. Using the counterexample domains from [9], one can construct domains in $\mathbb{C}^n$, $n \ge 3$, with the property that P_0 does not preserve smoothness up to the boundary, while $P_2, \dots, P_n$ do ([169], page 337).

We now show that the assumptions in Theorem 5.7 with $M = 1$ (which do not discriminate among form levels) imply those in Theorem 5.22 (at all form levels).

Proposition 5.26. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Assume the assumptions in Theorem 5.7 are satisfied, with $M = 1$. Then the assumptions in Theorem 5.22 hold for $1 \le q \le n$.*

Proof. By Lemma 5.23, we only have to consider the case $q = 1$. The arguments are similar to arguments in the proofs of Lemma 4 and Proposition 1 in [287]. Denote by ρ a defining function for Ω. Fix $\varepsilon > 0$. Define functions h_ε and vector fields Y_ε, (complex) tangential to the level sets of ρ, in a neighborhood U_ε of K by

$$X_\varepsilon = e^{-h_\varepsilon} L_n + Y_\varepsilon, \tag{5.133}$$

where $\{X_\varepsilon\}$ denotes the family of vector fields provided by Theorem 5.7. Here and throughout the proof we assume that U_ε has been shrunk if necessary so that everything is well defined. The family $\{h_\varepsilon\}$ satisfies (5.114) on U_ε, modulo rescaling in the sense of Remark (i) after the statement of Theorem 5.22. Further shrinking U_ε, we may assume that h_ε has an extension to a function in $C^\infty(\overline{\Omega})$, still denoted by h_ε, so that (5.114) holds for the family of the extended functions. Modulo rescaling, this family $\{h_\varepsilon\}$ will satisfy the assumption in Theorem 5.22. We need to verify (5.115).

Let $u = \sum_{j=1}^n u_j d\bar z_j \in C^\infty_{(0,1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar\partial^*)$. Choose a cutoff function $\varphi_\varepsilon \in C_0^\infty(U_\varepsilon)$ that is identically equal to one in a neighborhood of K. Then

$$\left\| \sum_{j,k=1}^n \frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j \partial \bar z_k} \frac{\partial \rho}{\partial \bar z_j} \overline{u_k} \right\|^2 \lesssim \int_\Omega \varphi_\varepsilon^2 \left| \sum_{j,k=1}^n \frac{\partial^2(e^{h_\varepsilon}\rho)}{\partial z_j \partial \bar z_k} \frac{\partial \rho}{\partial \bar z_j} \overline{u_k} \right|^2 + C_\varepsilon \|(1-\varphi_\varepsilon)u\|^2. \tag{5.134}$$

$(1 - \varphi_\varepsilon)u$ has support that meets the boundary inside the set of points of finite type (the complement of K). Therefore, the last term in (5.134) can be estimated, via interpolation of Sobolev norms and pseudolocal subelliptic estimates (see the discussion following (3.75); alternatively, if K is taken to be the set of weakly pseudoconvex points, use Theorem 3.6), by $\varepsilon(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) + C_\varepsilon \|u\|^2_{-1}$, which is what is required in Theorem 5.22. To estimate the main term in (5.134), note first that

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k}(e^{h_\varepsilon}\rho) = \frac{\partial}{\partial \bar{z}_k}\left(e^{h_\varepsilon}\frac{\partial\rho}{\partial z_j}\right) + \frac{\partial^2 e^{h_\varepsilon}}{\partial z_j \partial \bar{z}_k}\rho + \frac{\partial e^{h_\varepsilon}}{\partial z_j}\frac{\partial\rho}{\partial \bar{z}_k}. \tag{5.135}$$

The contributions from the second and third terms on the right-hand side of (5.135) are always under control, as follows. The contribution from the term containing the factor ρ is estimated as the corresponding term in (5.117). The contribution from the third term contains $\sum_{k=1}^{n}(\partial\rho/\partial\bar{z}_k)\overline{u_k}$, the conjugate of the normal component of u, and so is dominated, via Lemma 2.12 and again interpolation of Sobolev norms, by the right-hand side of (5.115). In order to control the contribution from the first term (the main term) on the right-hand side of (5.135), we invoke the commutator estimates (ii) for the family $\{X_\varepsilon\}$ in Theorem 5.7.

Denote by L_u the vector field $L_u = \sum_{j=1}^{n} u_j(\partial/\partial z_j)$. Then L_u is complex tangential at points of the boundary ($L_u\rho = 0$ at points of the boundary, because $u \in \mathrm{dom}(\bar{\partial}^*)$). (ii) in Theorem 5.7 gives

$$\left|\partial\rho(([X_\varepsilon, \overline{L_u}])\right| \lesssim \varepsilon|L_u| = \varepsilon|u| \quad \text{on } U_\varepsilon; \tag{5.136}$$

we have used that terms where X_ε acts on one of the $\overline{u_j}$'s disappear when $\partial\rho$ is applied. On the other hand, $\partial\rho([X_\varepsilon, \overline{L_u}])$ is related to the main term in (5.135) as follows:

$$\begin{aligned} e^{h_\varepsilon}\partial\rho\left([X_\varepsilon, \overline{L_u}]\right) &= \sum_{j,k=1}^{n} \overline{u_k}\frac{\partial}{\partial\bar{z}_k}\left(\frac{e^{-h_\varepsilon}}{|\partial\rho|^2}\frac{\partial\rho}{\partial\bar{z}_j} + (Y_\varepsilon)_j\right)\frac{\partial\rho}{\partial z_j}e^{h_\varepsilon} \\ &= \frac{-e^{-h_\varepsilon}}{|\partial\rho|^2}\sum_{j,k=1}^{n}\frac{\partial\rho}{\partial\bar{z}_j}\overline{u_k}\frac{\partial}{\partial\bar{z}_k}\left(e^{h_\varepsilon}\frac{\partial\rho}{\partial z_j}\right) - e^{h_\varepsilon}\sum_{j,k=1}^{n}(Y_\varepsilon)_j\overline{u_k}\frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}. \end{aligned} \tag{5.137}$$

We have used that $\sum_{j,k=1}^{n}\frac{e^{-h_\varepsilon}}{|\partial\rho|^2}\frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial\rho}{\partial z_j}e^{h_\varepsilon} \equiv 1$ and that $\sum_{j,k=1}^{n}(Y_\varepsilon)_j\frac{\partial\rho}{\partial z_j} \equiv 0$. (5.136) and (5.137) together with the uniform bounds on h_ε show that the contribution to the main term in (5.134) coming from the first term on the right-hand side of (5.135) satisfies

$$\begin{aligned} &\int_\Omega \varphi_\varepsilon^2\left|\sum_{j,k=1}^{n}\frac{\partial\rho}{\partial\bar{z}_j}\overline{u_k}\frac{\partial}{\partial\bar{z}_k}\left(e^{h_\varepsilon}\frac{\partial\rho}{\partial z_j}\right)\right|^2 \\ &\qquad \lesssim \varepsilon^2\|u\|^2 + \int_\Omega \varphi_\varepsilon^2\left|\sum_{j,k=1}^{n}(Y_\varepsilon)_j\overline{u_k}\frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}\right|^2. \end{aligned} \tag{5.138}$$

The term $\varepsilon^2\|u\|^2$ is dominated by the right-hand side of (5.115) (modulo rescaling). It remains to estimate the last term in (5.138). We have

$$\varphi_\varepsilon^2\Big|\sum_{j,k=1}^{n}(Y_\varepsilon)_j\overline{u_k}\frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}\Big|^2 \lesssim \varphi_\varepsilon^2\Big|\sum_{j,k=1}^{n}(Y_\varepsilon)_j\overline{(u_{\mathrm{Tan}})_k}\frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}\Big|^2 + \varphi_\varepsilon^2\Big|\sum_{j,k=1}^{n}(Y_\varepsilon)_j\overline{(u_{\mathrm{Norm}})_k}\frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}\Big|^2. \tag{5.139}$$

$\sum_{j,k=1}^{n}(\partial^2\rho/\partial z_j\partial\bar{z}_k)w_j\overline{w_k}$ is bounded from below by $-C\delta|w|^2$ on the complex tangent space to the boundary of the sublevel set $\{z\in\Omega\mid\rho(z)<-\delta\}$. Therefore, $\sum_{j,k=1}^{n}\big((\partial^2\rho/\partial z_j\partial\bar{z}_k)-C\rho\delta_{j,k}\big)\,w_j\overline{w_k}\geq 0$ (on the complex tangent space to the sublevel sets of ρ), and the Cauchy–Schwarz inequality applies to this Hermitian form. We use this observation to estimate the first term on the right-hand side of (5.139); this is analogous to the discussion immediately preceding (5.117). The result is that the right-hand side of (5.139) is dominated by

$$C_\varepsilon\Big(\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(\varphi_\varepsilon u_{\mathrm{Tan}})_j\overline{(\varphi_\varepsilon u_{\mathrm{Tan}})_k} - C\rho|\varphi_\varepsilon u_{\mathrm{Tan}}|^2\Big) + C^2\varphi_\varepsilon^2\rho^2\Big|\sum_{j=1}^{n}(Y_\varepsilon)_j\overline{(u_{\mathrm{Tan}})_j}\Big|^2 + |\varphi_\varepsilon u_{\mathrm{Norm}}|^2. \tag{5.140}$$

The last term in (5.140) arises from the last term in (5.139). The integrals over Ω of the first and the last term in (5.140) are now estimated like the corresponding terms were in (5.120) and in (5.119), respectively, with some extra care because the cutoff function φ_ε depends on ε. Specifically, the term $\|u\|^2$ in (5.122) has to be replaced by $\||\nabla\varphi_\varepsilon|u\|^2$. Similarly, the term $\||\nabla\varphi_\varepsilon|u\|^2$ has to be added into the right-hand side of (5.119). This term can be estimated in the same manner as the term $\|(1-\varphi_\varepsilon)u\|^2$ in (5.134). The integrals of the two terms in (5.140) that contain a factor ρ are estimated in the same way that the middle term in (5.117) was (note that $|u_{\mathrm{Tan}}|^2\leq|u|^2$), with the obvious modification to account for C_ε. The result is that the right-hand side of (5.139) is dominated by the right-hand side of (5.115). This completes the proof of Proposition 5.26. □

The above proof shows the connection between (5.115) and (ii) in Theorem 5.7: by (5.134) and the estimate for $\|(1-\varphi_\varepsilon)u\|$, we only have to control the portion over $\Omega\cap U_\varepsilon$ of the left-hand side of (5.115). There, modulo a factor that is bounded uniformly in ε and terms that are dominated by its right-hand side, the left-hand side of (5.115) is just the square of the $\mathcal{L}^2$-norm of $\varphi_\varepsilon\partial\rho([X_\varepsilon,\overline{L_u}])$. Then, if $\varphi_\varepsilon\partial\rho([X_\varepsilon,\overline{L_u}])$ is $O(\varepsilon)|u|$ on U_ε, its $\mathcal{L}^2$-norm on $\Omega\cap U_\varepsilon$ is a fortiori $O(\varepsilon)\|u\|\leq O(\varepsilon)(\|\bar\partial u\|+\|\bar\partial^* u\|)$. So (5.115) arises naturally as the 'correct' $\mathcal{L}^2$-version of the pointwise commutator condition in Theorem 5.7. That it also covers compactness (trivially) is an indication that this generalization may point in the right direction. In this regard, the reader should also consult Remark (i) following the proof of Theorem 5.22. We now give this proof. That is, we show that the $\mathcal{L}^2$-version of the commutator condition is still strong enough to imply global regularity.

Proof of Theorem 5.22. We keep the setup from the proof of Theorem 5.7. Recall that there we established Sobolev estimates for the Bergman projections; those for the $\bar\partial$-Neumann operators then followed from Theorem 5.5. Here we will similarly prove Sobolev estimates for P_m for $m \geq (q-1)$. Note that the downward induction on the degree q from the proof of Theorem 5.7 also applies here; this uses the fact that when the assumptions in Theorem 5.22 are satisfied at level q, then they are also satisfied at the levels $q+1, \dots, n$ (Lemma 5.23). The next step is to show a priori estimates; this is done via a relatively simple adaption of the proof of Theorem 5.7 (with $M = 1$). We only give the argument for these estimates in $W^1_{(0,q)}(\Omega)$. This will show the modifications that are needed; these modifications work in the case $q > 1$ as well.

If $\{h_\varepsilon\}$ is the family from Theorem 5.22, set

$$X_\varepsilon = e^{-h_\varepsilon} L_n. \tag{5.141}$$

Because of (5.114), the family $\{X_\varepsilon\}$ satisfies (i) in Theorem 5.7 (with $M = 1$). Now the argument follows that in the proof of Theorem 5.7 until (5.60). When $k = 1$, (5.60) shows that what needs to be estimated is

$$\begin{aligned} &\left|\left(\varphi(X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i L_n) N_q \bar\partial u, [\bar\partial, X_\varepsilon - \overline{X_\varepsilon} + b_\varepsilon i \overline{L_n}]\varphi P_{q-1} u\right)\right| \\ &\qquad + C_\varepsilon \left(\|u\|_1^2 + \|N_q \bar\partial u\| \|P_{q-1} u\|_1\right). \end{aligned} \tag{5.142}$$

Note that in contrast to (5.60), we may take the cutoff function φ to be independent of ε: h_ε is defined on all of $\overline{\Omega}$, and L_n is defined globally near the boundary. Using that $\|N_q \bar\partial u\| \|P_{q-1} u\|_1 \leq s.c \|P_{q-1} u\|_1^2 + \text{l.c.} \|N_q \bar\partial u\|^2$ shows that the term in the second line of (5.142) is benign. In the main term, we estimate the contributions coming from $\overline{X_\varepsilon}$, $\overline{L_n}$, and commutators of $\bar\partial$ with them, with Lemma 5.6 (replacing $P_{q-1}u$ by $u - P_{q-1}u$). Therefore, we are left with estimating

$$\begin{aligned} &\left|\left(\varphi(X_\varepsilon + b_\varepsilon i L_n) N_q \bar\partial u, [\bar\partial, X_\varepsilon]\varphi P_{q-1} u\right)\right| \\ &\qquad = \left|\left(\varphi(e^{-h_\varepsilon} + b_\varepsilon i) L_n N_q \bar\partial u, [\bar\partial, X_\varepsilon]\varphi P_{q-1} u\right)\right|. \end{aligned} \tag{5.143}$$

Here the argument departs from that in the proof of Theorem 5.7 in that we deal more carefully with the commutator $[\bar\partial, X_\varepsilon]$. Roughly speaking, rather than estimating each coefficient $([\bar\partial, X_\varepsilon]\varphi P_{q-1} u)_{jJ}$ individually, we will estimate, for each J, the inner product of $\sum_{j=1}^n ([\bar\partial, X_\varepsilon]\varphi P_{q-1} u)_{jJ}\, d\bar z_j$ with $\sum_{j=1}^n \varphi(L_n N_q \bar\partial u)_{jJ}\, d\bar z_j$. If $u = \sum'_{|J|=q-1} u_J\, d\bar z_J$, then

$$\begin{aligned} \bar\partial X_\varepsilon u - X_\varepsilon \bar\partial u &= \sum_{j,J}{}' \frac{\partial}{\partial \bar z_j}(X_\varepsilon u_J)\, d\bar z_j \wedge d\bar z_J - \sum_{j,J}{}' X_\varepsilon \frac{\partial u_J}{\partial \bar z_j}\, d\bar z_j \wedge d\bar z_J \\ &= \sum_{j,J}{}' \frac{\partial X_\varepsilon}{\partial \bar z_j} u_J\, d\bar z_j \wedge d\bar z_J = \sum_{j,k,J}{}' \frac{\partial}{\partial \bar z_j}\left(\frac{e^{-h_\varepsilon}}{|\partial\rho|^2}\frac{\partial \rho}{\partial \bar z_k}\right)\frac{\partial u_J}{\partial z_k}\, d\bar z_j \wedge d\bar z_J, \end{aligned} \tag{5.144}$$

where $\partial X_\varepsilon/\partial\bar{z}_j$ is understood coefficientwise, and the prime in the sums only relates to summation over J. Because tangential derivatives of type $(1,0)$ are under control by Lemma 5.6 (replacing $P_{q-1}u$ by $u - P_{q-1}u$), we only have to consider the normal component of $\partial/\partial z_k$ when substituting (5.144) into (5.143). This normal component is $(\partial\rho/\partial z_k)L_n$. Therefore, substituting into (5.143) shows that what we have to estimate is

$$\left|\left(\sum_{j,k,J}{}' \frac{\partial}{\partial\bar{z}_j}\left(\frac{e^{-h_\varepsilon}}{|\partial\rho|^2}\frac{\partial\rho}{\partial\bar{z}_k}\right)\frac{\partial\rho}{\partial z_k}L_n(\varphi P_{q-1}u)_J\,d\bar{z}_j\wedge d\bar{z}_J, \varphi(e^{-h_\varepsilon}+b_\varepsilon i)L_nN_q\bar{\partial}u\right)\right|. \tag{5.145}$$

The arguments in (5.137) and (5.135) show that (5.145) is dominated uniformly in ε by

$$\begin{aligned}&\left|\left(\frac{e^{-2h_\varepsilon}}{|\partial\rho|^2}\sum_{j,k,J}{}' \frac{\partial\rho}{\partial\bar{z}_k}\frac{\partial^2(\rho e^{h_\varepsilon})}{\partial\bar{z}_j\partial z_k}L_n(\varphi P_{q-1}u)_J\,d\bar{z}_j\wedge d\bar{z}_J, \varphi(e^{-h_\varepsilon}+b_\varepsilon i)L_nN_q\bar{\partial}u\right)\right|\\ &\quad+\left|\left(\frac{e^{-2h_\varepsilon}}{|\partial\rho|^2}\sum_{j,k,J}{}' \frac{\partial\rho}{\partial\bar{z}_k}\left(\frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial e^{h_\varepsilon}}{\partial z_k}+\rho\frac{\partial^2 e^{h_\varepsilon}}{\partial\bar{z}_j\partial z_k}\right)L_n(\varphi P_{q-1}u)_J\,d\bar{z}_j\wedge d\bar{z}_J,\dots\right.\right.\\ &\qquad\qquad\left.\left.\dots,\varphi(e^{-h_\varepsilon}+b_\varepsilon i)L_nN_q\bar{\partial}u\right)\right|.\end{aligned} \tag{5.146}$$

The contribution from the second line of (5.146) corresponds to the contribution that the last two terms in (5.135) make to (5.134), and the estimates are analogous. The term containing the factor ρ is estimated by $\varepsilon\|P_{q-1}u\|_1\|N_q\bar{\partial}u\|_1+C_\varepsilon\|P_{q-1}u\|_1\|N_q\bar{\partial}u\|_{1,V_\varepsilon}$, where $V_\varepsilon\subset\subset\Omega$. By (2.85) (interior elliptic regularity), the latter term is bounded by $C_\varepsilon\|P_{q-1}u\|_1\|\bar{\partial}^*N_q\bar{\partial}u\|\le C_\varepsilon\|P_{q-1}u\|_1\|u\|$. For the term containing the factor $\partial\rho/\partial\bar{z}_j$, some extra care is needed. When j and J are fixed in (5.146), the term $d\bar{z}_j\wedge d\bar{z}_J$ in the inner product picks out the coefficient $(\cdot)_{jJ}$ of the form on the right-hand side. Summation over j therefore produces

$$\sum_{j=1}^n\frac{\partial\rho}{\partial\bar{z}_j}\overline{(L_nN_q\bar{\partial}u)_{jJ}}=\overline{L_n}\left(\sum_{j=1}^n\frac{\partial\rho}{\partial\bar{z}_j}\overline{(N_q\bar{\partial}u)_{jJ}}\right)-\sum_{j=1}^n\overline{L_n}\left(\frac{\partial\rho}{\partial\bar{z}_j}\right)\overline{(N_q\bar{\partial}u)_{jJ}}. \tag{5.147}$$

We have used here that L_n acts coefficientwise in the Euclidean chart. The last term in (5.147) contains no derivatives of $N_q\bar{\partial}u$ and so poses no problem. The expression in parentheses in the first term on the right-hand side of (5.147) is the conjugate of a coefficient of the normal component $(N_q\bar{\partial}u)_{\text{norm}}$. The $\mathcal{L}^2$-norm of this term is therefore dominated by $\|(N_q\bar{\partial}u)_{\text{norm}}\|_1$, which in turn is estimated by $\|\bar{\partial}^*N_q\bar{\partial}u\|\le\|u\|$, in view of Lemma 2.12. Its contribution in (5.146) is thus dominated by $C_\varepsilon\|P_{q-1}u\|_1\|u\|$.

The remaining term in (5.146) is the main term. It is here that estimate (5.115) in the assumptions in Theorem 5.22 enters into the argument. Upon writing out the inner product and rearranging factors, the term becomes

$$\left|\int_{\Omega}\frac{e^{-2h_\varepsilon}}{|\partial\rho|^2}\sum_J{}'\Big(\sum_{j,k=1}^{n}\frac{\partial\rho}{\partial\bar{z}_k}\frac{\partial^2(\rho e^{h_\varepsilon})}{\partial\bar{z}_j\partial z_k}\overline{(\varphi L_n N_q\bar{\partial}u)_{jJ}}\Big)\overline{(e^{-h_\varepsilon}+b_\varepsilon i)}L_n(\varphi P_{q-1}u)_J\,dV\right|$$
$$\lesssim\left\|\sum_J{}'\Big(\sum_{j,k=1}^{n}\frac{\partial\rho}{\partial\bar{z}_k}\frac{\partial^2(\rho e^{h_\varepsilon})}{\partial\bar{z}_j\partial z_k}\overline{(\varphi L_n N_q\bar{\partial}u)_{jJ}}\Big)d\bar{z}_J\right\|\|P_{q-1}u\|_1. \tag{5.148}$$

We have used the uniform bounds on e^{-h_ε} and on b_ε. The first factor in the second line of (5.148) is the left-hand side of (5.115) in Theorem 5.22, with u replaced by $\varphi L_n N_q\bar{\partial}u$. However, we cannot yet apply the theorem, because $L_n N_q\bar{\partial}u$ need not be in the domain of $\bar{\partial}^*$. But the remedy is the usual one. We first replace L_n by $L_n-\overline{L_n}$ to obtain a tangential operator. This makes an error whose $\mathcal{L}^2$-norm is bounded by $C_\varepsilon\|\varphi\overline{L_n}N_q\bar{\partial}u\|\le C_\varepsilon(\|\bar{\partial}^*N_q\bar{\partial}u\|+\|N_q\bar{\partial}u\|)\le C_\varepsilon\|u\|$ (by Lemma 5.6). The resulting error in (5.148) is therefore dominated by $C_\varepsilon\|u\|\|P_{q-1}u\|_1$, and so is benign. Then we let $L_n-\overline{L_n}$ act in special boundary charts (via a partition of unity). This makes an error that does not involve derivatives of $\varphi L_n N_q\bar{\partial}u$, and which is therefore also bounded by $C_\varepsilon\|u\|\|P_{q-1}u\|_1$. Now we can apply (5.115). This gives that the square of the (modified) first factor in the second line of (5.148) is dominated by

$$\begin{aligned}&\varepsilon\big(\|\bar{\partial}(\varphi(L_n-\overline{L_n})N_q\bar{\partial}u)\|^2+\|\bar{\partial}^*(\varphi(L_n-\overline{L_n})N_q\bar{\partial}u)\|^2\big)\\&\qquad+C_\varepsilon\|\varphi(L_n-\overline{L_n})N_q\bar{\partial}u\|_{-1}^2\\&\lesssim\varepsilon\big(\|N_q\bar{\partial}u\|_1^2+\|\varphi(L_n-\overline{L_n})\bar{\partial}^*N_q\bar{\partial}u\|^2\big)+C_\varepsilon\|N_q\bar{\partial}u\|^2\\&\lesssim\varepsilon\big(\|N_q\bar{\partial}u\|_1^2+\|P_{q-1}u\|_1^2+\|u\|_1^2\big)+C_\varepsilon\|u\|^2.\end{aligned} \tag{5.149}$$

To obtain the first inequality, we have commuted $\bar{\partial}$ and $\bar{\partial}^*$ with $\varphi(L_n-\overline{L_n})$. The second inequality then follows by writing $\bar{\partial}^*N_q\bar{\partial}u$ as $u-P_{q-1}u$.

Collecting all our estimates, absorbing terms, etc., we recover (5.63):

$$\|P_{q-1}u\|_1^2\le\varepsilon C\left(\|P_{q-1}u\|_1^2+\|N_q\bar{\partial}u\|_1^2\right)+C_\varepsilon\|u\|_1^2, \tag{5.150}$$

where the constant C does not depend on ε. The argument thus returns to that in the proof of Theorem 5.7: inserting (5.51) into (5.150) and choosing ε small enough gives the desired a priori estimate

$$\|P_{q-1}u\|_1^2\le C\|u\|_1^2. \tag{5.151}$$

It remains to turn these a priori estimates into genuine estimates. Here, a substantial difference occurs compared to the proof of Theorem 5.7. There, we exhausted Ω by strictly pseudoconvex domains, and then argued that the a priori estimates were uniform on the approximating subdomains. Because the Bergman projections on these domains were known to preserve smoothness up to the boundary, these estimates had to be genuine estimates (starting with a form smooth on the closure), and a limiting process then gave the estimates on Ω. By contrast, the $\mathcal{L}^2$-estimate (5.115) does not seem to be strong enough to be inherited by the approximating subdomain (with uniform constants). Consequently, we have to use another regularization method. A natural one to use is elliptic regularization (since we know that it works in a special case, that of compactness). It turns out that this introduces some technical difficulties which force a certain reformulation of the proof given for the a priori estimates. This is because the argument does not rely solely on properties of the form $Q(u,u)$ that hold equally for the regularized form $Q_\delta(u,u)$ (see (3.18)). We do not reproduce the details here, but instead refer the reader to [287]. $\square$

Remarks. (i) The flavor of the condition (5.115) in Theorem 5.22 is potential theoretic. That is not surprising: global regularity is not likely to be determined by geometric conditions (unlike the much stronger property of subellipticity). But it is interesting to extract a geometric sufficient condition for global regularity from (5.115), and to see what this condition turns out to be. Let $q = 1$ for simplicity. Then the quantity on the left-hand side of (5.115) is comparable to the square of the $\mathcal{L}^2$-norm of a function of the following kind: the mixed (complex tangential unit vector - complex normal unit vector) term in the complex Hessian of $e^{h_\varepsilon}\rho$ times $|u|$. (More, precisely, one has to choose local bases of $T^{(1,0)}(b\Omega)$ and express u in these bases.) The estimate $\|u\|^2 \lesssim \|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2$ suggests that in order to achieve (5.115), this mixed term should be a multiplier in $\mathcal{L}^2(\Omega)$ of operator norm not exceeding $\sqrt{\varepsilon}$. This operator norm coincides with the sup-norm of the multiplier. So this multiplier should be uniformly bounded by $\sqrt{\varepsilon}$. Since we have control over compactly supported terms, this estimate is actually only needed at (hence near) the boundary. (5.135) and (5.137) in the proof of Proposition 5.26 show that this happens precisely when $|\partial\rho([e^{h_\varepsilon}L_n, \bar{L}])(P)| \lesssim \sqrt{\varepsilon}|L(P)|$ for a complex tangential field L. In other words, we recover the vector field method (Theorems 5.7 and 5.9). In this sense, the vector field method constitutes the geometric content of Theorem 5.22.

(ii) Just as in the case of Theorem 5.9, there is a version of Theorem 5.22 that is analogous to Theorem 5.7 in that there are finitely many families $\{h_{m,\varepsilon}\}$, $1 \le m \le M$, where for each m, the lower bound on $|e^{h_{m,\varepsilon}}|$ (uniform in ε) is only imposed on a portion of the boundary. The proof is the same (but using the modification for $M > 1$ in the proof of Theorem 5.7). Inspection of the proof of Proposition 5.26 reveals that the uniform transversality of $\{X_\varepsilon\}$ is only used to get a uniform lower bound on $\{h_\varepsilon\}$. Accordingly, this proof shows that the assumptions in the general version of Theorem 5.9 imply those in the general version of Theorem 5.22.

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